INTERMEDIATE ALGEBRA

DeLURY

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BY

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PREFACE

Advantage has been taken of the issue of a new edition of the Intermediate Algebra to revise the text and to make a number of changes.

To meet the wishes of teachers who have used, or propose to use, the book in the advanced classes of the secondary schools, additions have been made in order that the prescribed courses may be formally covered. To the chapter on equations there have been added exercises bearing on or developing further the theory. A chapter on Scales of Notation has been introduced. The note on Annuities, incidental to the Geometrical Progression, has been expanded to constitute a chapter in which are considered the simpler problems of finance related to annuities, and the use of the fundamental tables of Interest and Annuities provided for and explained.

The Miscellaneous Examples that were given in the earlier edition have been retained in the hope that the more adventurous students, in particular candidates for Honours at Matriculation, may find in them a help and a stimulus.

The chapter on Exponential and Logarithmic Series has been enlarged in order to give prominence to the concrete problem of the construction of tables of logarithmics, it being felt that in this way the significance of the theory is best brought out.

ALFRED T. DELURY.

TORONTO, July 15, 1918.

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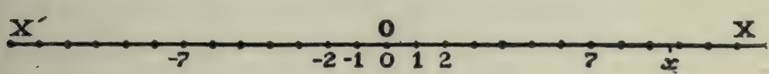
CHAPTER I

EQUATIONS

1. Preliminary. In the expression $2x - 3$ appear the numbers $2, -3, x$. Of these, $2, -3$, having definite values which cannot change, are called **constants,** while the number x, regarded as capable of assuming different values, is called a **variable.** In general, it is supposed that the variable x may assume, or pass through, all assignable positive or negative values, as well as the value zero. Numbers generally present themselves as measuring some physical or geometrical quantity, the constant measuring a certain definite quantity and the variable measuring a quantity undergoing change. On this account numbers and expressions involving numbers are frequently spoken of as quantities. It is to be noted, too, that, in the case of variable quantities, the physical or geometrical conditions may be such as to limit the range of variation of the variable.

A convenient and useful representation of number, whether constant or variable, is found in the straight line regarded as indefinitely produced each way. A point O is taken, in this line, as origin, or place from which measurements are made.



Each point on this line, being at a certain distance from the origin, can be looked upon as carrying a certain number, namely, the measure of the distance of that point from the origin. It is agreed that positive numbers are to be carried by points to the right of the origin, and negative numbers by points to the left, while the number zero is carried by the point O, the origin. A fixed point on the line represents a constant number; a point, moving or regarded as capable of moving in the line, a variable number.

Return now to the expression $2x-3$. If to x be assigned the values

$$0,\ \tfrac{1}{2},\ 1,\ 1\tfrac{1}{2},\ 2,\ \ldots\ldots$$

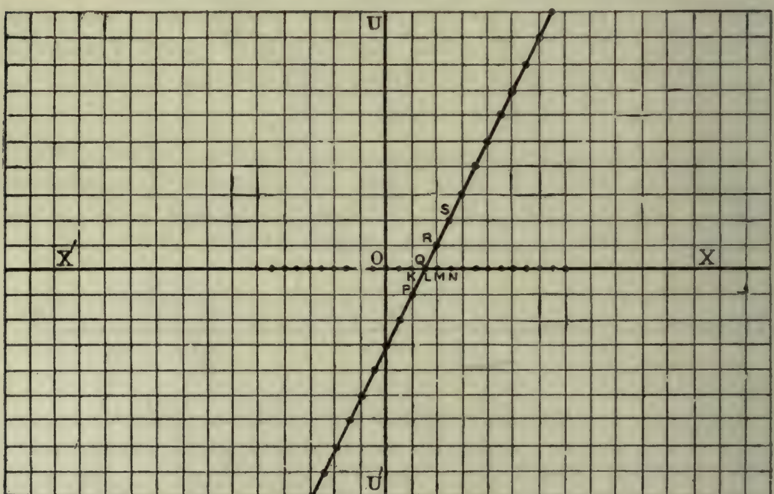
the values of $2x-3$ are found to be

$$-3,\ -2,\ -1,\ 0,\ 1,\ \ldots\ldots$$

It follows then that $2x-3$ is also a variable.

Any quantity whose value is determined when the value of a certain variable quantity, as x, is given, is called a **function** of that variable. Thus, $2x-3$ is a function of x. This function is further described as **one-dimensional,** or **linear** in the variable x.

As has been seen, the linear function $2x-3$ is a variable, its value depending on and changing with that of x. On this account x is called the **independent variable,** and if the value of the function $2x-3$ is denoted by u, then u is called the **dependent variable.** The relation of u to x can be exhibited by means of a diagram. First,



draw X′OX on which to represent the values of x, O being the origin. Then through O draw UOU′ at right angles to this line.

First, suppose x to have the value $2\frac{1}{2}$, carried by the point N. The corresponding value of u is found to be 2. Mark the point S at a

distance 2 from N in the direction OU. Then this point S indicates by its distance from the point N in OX, just below it, the value of u for $x = 2\frac{1}{2}$, the value carried by N.

Next, for $x = 2$, carried by the point M, the value of u is 1, and, as before, the point R indicates by its position that the value of u is 1 if $x = 2$.

So for $x = 1\frac{1}{2}$, carried by the point L, the value of u is 0, and the point Q (the same point as L) on the line X′OX indicates this fact.

For $x = 1$, carried by the point K, the value of u is -1. We agree to indicate negative values of u by measurements made below the line X′OX, so that the point P indicates that $u = -1$ for $x = 1$.

Thus, to each value of x corresponds a value of u, and if we construct the assemblage of points PQRS ..... giving the values of u corresponding to the values of x belonging to all the points on X′OX, a continuous line PQRS ..... will be formed. Here this continuous line is found to be a straight line. It is said to be the **graph** of the function $2x - 3$, and, as has been explained, it puts in evidence the value of the function for any or all values of x considered. The lines X′OX, UOU′, to which the measurements determining the graph have been referred, are called the **axes.**

Consider now the expression $ax + b$. Here it is supposed that a and b are constants, and that x is a variable, so that $ax + b$ is a linear function of x. The numbers a and b may have any values, but, whatever be these values, they are constant. The expression $ax + b$ is thus the **general linear function** of x. For any given values of a and b the graph of $ax + b$ may be constructed, and can be seen to be a straight line.

An expression as

$$\frac{2x - 3}{3x - 5}$$

in accordance with what has been stated, is a function of x. It is also a linear function of x as in it x appears in the first and no higher degree. Here, however, x occurs in the denominator, or, in other words, the function is fractional in its relation to x. On this account it is said to be a **fractional linear function** of x, while a function as $ax + b$ is said to be an **integral linear function.**

EXERCISES

1. Construct the graph of the following linear functions:

$$x-7,\ 3x,\ 2x+4,\ 6-8x.$$

2. Construct the graph of

$$\frac{x+3}{2x-1}$$

for values of x in the interval $(-2, +2)$.

3. Shew that, whatever be the values of a and b, the graph of $ax+b$ is a straight line.

Hence shew that when two values of any integral linear function have been found the graph may be constructed.

2. **Significance of an Equation.** Suppose that the solution of the following simple problem is required:

Find the number, the double of which diminished by 3 is equal to zero.

Let x denote the number. Then by the condition given

$$2x-3=0.$$

We have here an **equation,** a statement to the effect that the expression $2x-3$ *is equal to* zero. It is readily seen that

$$2x=3 \text{ and } x=1\tfrac{1}{2}.$$

The rôle of the equation is now manifest. If it had been a question of the *expression* or *function* $2x-3$ only, we should have regarded x as a variable, and as x varied the value of $2x-3$ also would have varied. The *equation* declares that $2x-3$ is not to vary but to have the definite value zero, and it follows that x is not a variable but has a determinate value $1\frac{1}{2}$. In an equation then, as the one just treated, the x, which is not a variable but a definite number whose value is sought, will be called the **unknown quantity,** or simply the **unknown.** The value of the unknown is called the **root** of the equation.

The relation of the *expression* $2x-3$ to the *equation* $2x-3=0$ is now clear. For different values of x the expression takes different values; the equation makes it impossible for x to vary, and we are required to find the invariable or definite value of x which will make $2x-3$ equal to zero. In like manner we could find the value of x that

would make $2x - 3$ equal to any given number. The graph constructed in the preceding Art. puts this in evidence, as well as the fact that when $2x - 3 = 0$, the value of x is $1\frac{1}{2}$.

Not infrequently there present themselves equalities which are not equations like the one considered, in that they do not require that x should have a definite value, but hold for all values of x. Such, for example, is the relation

$$(x+1)(x-1) - (x^2 - 1) = 0$$

or, which is the same thing,

$$(x+1)(x-1) = x^2 - 1.$$

If the implied multiplication is performed it is seen that the equality holds whatever be the value of x, or, in other words, that $(x+1)(x-1)$ and $x^2 - 1$ are two *different forms of one expression.* Such an equality is generally called an **identical equation** or an **identity,** so that when the term equation is employed it is to be understood in the sense first explained.

EXERCISES

1. Construct the equations pertaining to the following problems :

(1) Find two consecutive integers such that their product will exceed the square of the less by 13.

(2) A father is three times as old as his son, but in 15 years he will be only twice as old as his son. Find their present ages.

(3) A can run 100 yd. in 10 sec., and B in 11 sec. They start together in a race of 100 yd. At the end of what time will A be midway between B and the winning post?

2. Shew that the following problems lead to identities, stating in each case the inference to be drawn from this fact :

(1) Find three consecutive integers such that the product of the greatest and the least is less than the square of the mean integer by unity.

(2) Divide a straight line of given length a (units), so that the square on the given line may be equal to the squares on the two parts, increased by twice the rectangle contained by the parts.

3. **The Simple Equation.** The general linear equation is

$$ax+b=0.$$

It is supposed that a is not zero; for if this were the case, then would ax equal zero, and, therefore, also b equal zero, and there would be no equation.

It follows at once that

$$ax=-b$$

and, therefore, that

$$x=-\frac{b}{a},$$

since, a not being zero, division is possible.

These results are necessary and it follows that:

Every linear equation has one and only one root.

It is well to suppose any such equation to have originated in some actual problem, x measuring some quantity whose magnitude is sought.

An equation of the form

$$ax+b=cx+d,$$

is readily seen to be not more general than the equation $ax+b=0$.

EXERCISES

1. Solve the equations:

(1) $\frac{2x-7}{5}+\frac{x-5}{6}=\frac{x+3}{7}+\frac{x+7}{9}.$

(2) $5x-\left(\frac{x}{2}-\frac{x}{3}\right)=7\left(\frac{x}{3}+\frac{x}{4}\right)+9.$

(3) $\frac{x}{p}+\frac{x}{q}+\frac{x}{r}=qr+rp+pq.$

(4) $\frac{x}{bc}+\frac{x}{ca}+\frac{x}{ab}=a+b+c.$

2. The sides of a triangle are 39 ft. 42 ft., 45 ft. in length. A perpendicular is drawn to the side whose length is 42 ft. from the opposite angle. Find the length of the segments into which this side is divided.

Construct the triangle according to scale, draw the perpendicular as indicated, and measure the segments to test the accuracy of the *drawing*.

4. The Quadratic Expression. Consider the expression

$$x^2 - 4x + 3.$$

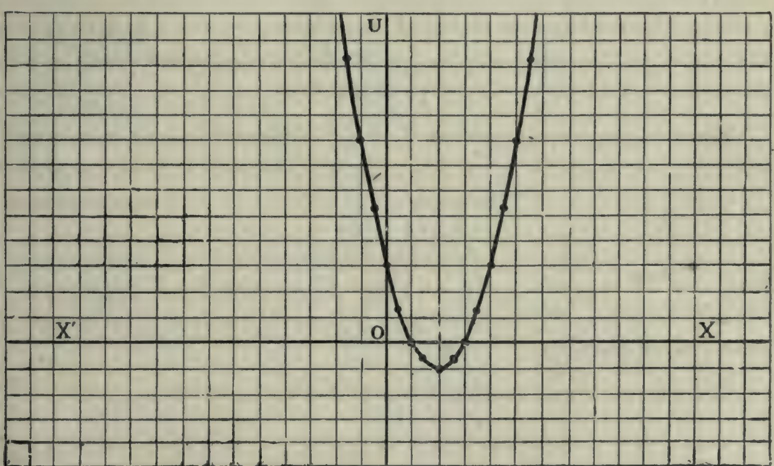
In this expression, the numbers -4, 3 are constants and the number x a variable. To each value of x corresponds a value of the expression. Thus, if x is assigned the values

$$-1,\ 0,\ 1,\ 2, \ldots\ldots$$

the values of $x^2 - 4x + 3$ are found to be

$$8,\ 3,\ 0,\ -1, \ldots\ldots$$

The expression $x^2 - 4x + 3$, for reasons already given, is then a function of x. It is further described as a **function of the second degree,** or a **quadratic function** as in it x occurs to the second power, and not to any higher power. The function is integral since x does not appear as a denominator or as making part of a denominator. If the value of the function is denoted by u, then u is a variable, and a graph may be constructed which will exhibit the dependence of u upon x.



The graph is not a straight line, as in the case of the linear function, but a curve which is called a **parabola.** It is to be noted that the curve is symmetrical about a line at right angles to the axis $X'OX$

through the point carrying the value 2 of x. The reason for this may be shewn, for we have

$$\begin{aligned} u &= x^2 - 4x + 3 \\ &= x^2 - 4x + 4 - 1 \\ &= (x-2)^2 - 1. \end{aligned}$$

Now, first let $x = 2 - k$, a value k less than, or behind **2.**

Then for this value of x

$$\begin{aligned} u &= (2 - k - 2)^2 - 1 \\ &= k^2 - 1. \end{aligned}$$

Next, let $x = 2 + k$, a value k greater than, or in advance of **2.**

Then for this value of x

$$\begin{aligned} u &= (2 + k - 2)^2 - 1 \\ &= k^2 - 1. \end{aligned}$$

Therefore, whatever be the value of k, these two values of u are the same, and hence any two values of x equidistant from 2 give the same value of u, or, in other words, the graph is symmetrical about a line through $x = 2$ at right angles to X′OX.

Further, the graph indicates that the smallest value, or **minimum,** of u is -1, and this when $x = 2$. The reason for this is manifest, for we have

$$u = (x-2)^2 - 1.$$

Now, $(x-2)^2$ is positive for all values of x except $x = 2$, when $(x-2)^2$ is equal to zero. When, therefore, x is equal to 2, the value of u is the smallest possible.

The general linear quadratic function may be written

$$ax^2 + bx + c$$

and for any assigned values of the constants a, b, c, the behaviour of the function under the variation of x may be studied, and indicated by a diagram.

EXERCISES

1. Construct the graphs of the following functions :

$$x^2,\ x^2-1,\ x^2-x+6,\ x^2-2x+3.$$

In each case find the minimum of the function and the value of x which gives this minimum.

2. Construct the graphs of the following functions :

$$-x^2+x-6,\ -x^2,\ -x^2+1,\ 3+2x-x^2.$$

In each case find the maximum of the function and the value of x which gives this maximum.

3. Construct the graphs of the following functions :

$$2x^2-3x-5,\ 3x^2-5x+7,\ 4+5x-2x^2.$$

In each case find the minimum or maximum of the function and the corresponding value of x.

4. Construct the graph of the fractional quadratic function

$$\frac{x^2-x-2}{x^2-5x+6}$$

for values of x between -2 and $+4$.

5. **Quadratic Equations.** Suppose that the solution of the following problem is required :

Divide the number 4 into two parts such that the sum of the parts exceeds their product by 1.

If one part be the number x, then the other part is $4-x$.

The sum of the parts $=x+(4-x)=4$.

The product of the parts $=x\ (4-x)$.

$$\therefore\ 4-x\ (4-x)=1$$
$$\therefore\ x^2-4x+4\quad =1$$
$$\therefore\ x^2-4x+3\quad =0.$$

In this **quadratic equation,** x is presumably not a variable but some definite number, an unknown as yet. We are to seek the value of x in this equation, or, in other words, to solve the equation.

If now

$$x^2 - 4x + 3 = 0$$

then must

$$(x-3)(x-1) = 0$$

and conversely. This last relation is satisfied if either

(1) $x - 3 = 0$, *i.e.*, $x = 3$; for then $x - 3 = 0$ and $x - 1 = 2$, so that the product $(x-3)(x-1)$ is equal to zero,

or

(2) $x - 1 = 0$, *i.e.*, $x = 1$, for then $x - 1 = 0$, and $x - 3 = -2$, so that the product $(x-3)(x-1)$ is equal to zero.

Thus the equation has *two roots*, namely 3 and 1, and it follows that x does not represent a definite number but one of two definite numbers. This is a result that might have been expected, since, x standing for one of the two parts, and the equation having been formed on this supposition, each part has an equal claim to recognition.

The root $x = 3$ gives 3 and 1 as the two parts of 4, and the root $x = 1$ gives 1 and 3 as those parts, so that the two solutions refer to one mode of division, the parts being mentioned in different orders.

Further, no value of x other than 3 and 1 will satisfy the equation, for any such value would make neither $x - 3$ nor $x - 1$ equal to zero and the product $(x-3)(x-1)$ could not then be zero. Thus, this quadratic equation has two and only two roots. It is to be noted, too, that the solution of this quadratic equation consists in replacing it by *two linear equations*.

The graph of the function given in Art. 3 brings out the fact that for $x = 1$ or $x = 3$ the value of the function $x^2 - 4x + 3$ is zero.

As in the case of the linear equation it is well to regard any proposed quadratic equation as having originated in some problem, x denoting some unknown number whose value is sought.

We shall now study the general quadratic equation with a view to discover general properties of such equations. Let the general quadratic equation be

$$ax^2 + bx + c = 0$$

where a is supposed not to be zero.

Then must

$$a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=0$$

$$\therefore\ a\left[\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)-\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)\right]=0$$

$$\therefore\ a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2\right]=0$$

$$\therefore\ a\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)=0$$

$$\text{or } a\left(x+\frac{b+\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b-\sqrt{b^2-4ac}}{2a}\right)=0.$$

This last equation is then satisfied by those values of x which satisfy the proposed equation and conversely. Now, in order that this last equation may be satisfied either

(1) $x+\dfrac{b+\sqrt{b^2-4ac}}{2a}$ *must* equal zero, which requires that

$$x=\frac{-b-\sqrt{b^2-4ac}}{2a},$$

or

2) $x+\dfrac{b-\sqrt{b^2-4ac}}{2a}$ *must* equal zero, which requires that

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a},$$

Plainly any value of x other than these will give to the factors on the left a value different from zero, so that the expression on the left cannot vanish. Accordingly,

The general quadratic equation has two and only two roots.

From the preceding it is plain that *to resolve into factors the expression* ax^2+bx+c and *to solve the equation* $ax^2+bx+c=0$ are equivalent problems. The quadratic expression can always be presented as the product of two factors linear in x and, it may be, an additional factor not involving x; the quadratic equation is solved by replacing it by two linear equations. This relationship of the expression and the equation may be brought out otherwise, as follows.

Denote the roots of the equation $ax^2+bx+c=0$ by m and n so that, say,

$$\left.\begin{aligned} m &= \frac{-b+\sqrt{b^2-4ac}}{2a} \\ n &= \frac{-b-\sqrt{b^2-4ac}}{2a} \end{aligned}\right\} \quad \text{I.}$$

These roots are algebraically irrational, that is, they require for their expression root symbols, though it may be that for given values of a, b, c, the arithmetical values of the roots may be found. If the roots are added and multiplied we have

$$\left.\begin{aligned} m+n &= -\frac{b}{a} \\ mn &= \frac{c}{a} \end{aligned}\right\} \quad \text{II.}$$

The relations I and II are equivalent; in I the values of the roots are explicitly given, but the results given in II, on account of their simple form, are often more useful. We now have the expression

$$ax^2+bx+c=a\left\{x^2+\frac{b}{a}x+\frac{c}{a}\right\}, \text{ (identically.)}$$

$$=a\left\{x^2-(m+n)x+mn\right\},$$

by II, m and n denoting the roots of the *equation* $ax^2+bx+c=0$.

$$\therefore\ ax^2+bx+c=a(x-m)(x-n)$$

as seen before.

The two roots of the general equation are, in general, different, the difference being

$$m-n=\frac{\sqrt{b^2-4ac}}{a}.$$

This difference will be zero, *i.e.*, the two roots will be equal should it be that $\sqrt{b^2-4ac}=0$, *i.e.*, $b^2-4ac=0$ or $b^2=4ac$.

In this case the solution of the equation would have been as follows:

$$ax^2 + bx + c = 0$$

$$\therefore\ a\left\{x^2 + \frac{b}{a}x + \frac{c}{a}\right\} = 0$$

$$\therefore\ a\left\{\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \left(\frac{b^2 - 4ac}{4a^2}\right)\right\} = 0$$

$$\therefore\ a\left\{x + \frac{b}{2a}\right\}^2 = 0, \text{ since } b^2 - 4ac = 0.$$

The expression $ax^2 + bx + c$ is then a perfect square, except for the constant factor a, *i.e.*, is a perfect square in its relation to x, and the equation $ax^2 + bx + c = 0$ has *two equal roots*, each being $-\dfrac{b}{2a}$.

The solution of the quadratic equation leads to two remarkable extensions of the number concept. The numbers first employed were those that came to be called integers, as 1, 2, 3, ... The operation of division, arising through the solution of the linear equation, led to fractional numbers, as $\frac{2}{3}$, $\frac{3}{4}$, ... Also, algebra, dealing with general numbers, led naturally to the introduction of negative numbers as distinguished from positive numbers. We are now led further to extend the domain of number.

For first, consider the equation,

$$x^2 + 3x - 1 = 0.$$

The roots are found to be

$$\frac{-3 + \sqrt{13}}{2} \text{ and } \frac{-3 - \sqrt{13}}{2}.$$

The square root of 13 cannot be exactly found, *i.e.*, cannot be expressed as a positive or negative integer or fraction, though it may be found to any degree of accuracy. Also, a line or other magnitude can be constructed which would have for measure the square root of 13. We are thus led to call $\sqrt{13}$ a number, an **irrational number**, in contrast to positive or negative integers or fractions which are **rational numbers.** Indeed, it is only after admitting the square root of 13 to be a number that we can say that the equation $x^2 + 3x - 1 = 0$ has two roots.

Next, consider the equation

$$x^2 - 2x + 5 = 0.$$

The roots are found to be

$$\frac{2 \pm \sqrt{-16}}{2}.$$

The symbol $\sqrt{-16}$ is a symbol of something for which no equivalent exists among the numbers already treated, for the square of any positive number or of any negative number is a positive number, and, therefore, cannot equal -16. After the analogy of the earlier extensions, we shall call $\sqrt{-16}$ a number, an **imaginary number,** to distinguish it from those numbers previously considered which will be called **real numbers.** These imaginary numbers are admitted as an outcome of the laws of algebra, and they will be supposed to enter into all operations in accordance with these laws. Thus, we say

$$\sqrt{-16} = \sqrt{16 \times -1} = \sqrt{16} \times \sqrt{-1} = \pm 4\sqrt{-1},$$

and the roots of the equation $x^2 - 2x + 5 = 0$ are

$$1 \pm 2\sqrt{-1}.$$

Also, it is only after admitting imaginary numbers that we can say that the equation treated has roots.

A number, as $1 + 2\sqrt{-1}$, in which one part is real and the other imaginary, is generally called a **complex number.**

It is readily seen that: *The roots of the equation* $ax^2 + bx + c = 0$ *will be*

(1) *real and unequal, if* $b^2 > 4ac$,

(2) *real and equal,* *if* $b^2 = 4ac$,

(3) *complex,* *if* $b^2 < 4ac$,

it being supposed that a, b, c, *the given constants of the equation, are real.*

EXERCISES

1. Resolve into linear factors :

$$x^2+7x-5\,; \quad 2x^2-7x+5\,;$$
$$3-5x-4x^2\,; \quad 7x^2-11x+5.$$

2. Solve the equations :

$$x^2-7x+3=0\,; \quad 3x^2-4x-5=0\,;$$
$$7-3x-5x^2=0\,; \quad 3x^2-11x+21=0.$$

3. Without solving the equations, determine the character of the roots—whether real or imaginary, and if real whether positive or negative—of the following :

$$x^2-9x+3=0\,; \quad 2x^2-7x-5=0\,;$$
$$2x^2+5x+1=0\,; \quad 3x^2-15x+10=0.$$

4. Construct the equation whose roots exceed those of the equation $x^2-7x+3=0$ by 1.

5. Construct the equation whose roots exceed those of the equation $ax^2+bx+c=0$ by h.

6. Construct the equation whose roots are twice those of the equation $x^2-5x+3=0$.

7. Construct the equation whose roots are m times those of the equation $x^2+px+q=0$.

8. Shew that each root of the equation $hx^2+2kx+h=0$ is the reciprocal of the other.

9. If m and n are the roots of the equation $2x^2-7x+3=0$ construct the equation whose roots are

$$(1)\ m^2 \text{ and } n^2\,; \quad (2)\ \frac{1}{m} \text{ and } \frac{1}{n}\,; \quad (3)\ m^2n \text{ and } mn^2.$$

10. If m and n are the roots of the equation $ax^2+bx+c=0$ construct the equation whose roots are

$$(1)\ m^2 \text{ and } n^2\,; \quad (2)\ \frac{1}{m} \text{ and } \frac{1}{n}\,; \quad (3)\ \frac{1}{m^2} \text{ and } \frac{1}{n^2}\,;$$

$$(4)\ m^3 \text{ and } n^3\,; \quad (5)\ m^2n \text{ and } mn^2\,; \quad (6)\ \frac{m}{n} \text{ and } \frac{n}{m}\,.$$

11. Find the condition that the equations $x^2+px+q=0$ and $x^2+rx+s=0$ may have a common root.

12. Find the conditions that the two equations

$$ax^2+bx+c=0\,; \quad px^2+qx+r=0$$

may have the same roots.

13. Shew that the value of the expression x^2-3x+2 is *zero* for $x=1$ or $x=2$, is *negative* for all values of x intermediate to 1 and 2, and is *positive* for all values of x exterior to the interval (1, 2).

14. Find for what values of x the expression x^2-x-12 is zero, for what values of x it is negative and for what values of x it is positive.

15. If the roots of the equation $x^2+px+q=0$ are known to be real, shew that the expression x^2+px+q is positive for values of x which are not roots and which do not lie between the roots.

If the roots are known to be equal, or if they are known to be imaginary, what can be inferred as to sign of the expression x^2+px+q?

16. Divide a given straight line of length a (units) so that the rectangle contained by the whole and one part may be equal to the square on the other part.

Interpret the roots in Euclid's construction.

6. **Equations of Degree Higher than the Second.** While the general equation of the third degree and that of the fourth degree admit solution it is not proposed to consider these solutions. However, there will be discussed certain examples of special types which do not require for their solution a knowledge of equations of degree higher than the second. The general equation of the fifth or of any higher degree does not admit algebraic solution.

It is to be remarked that the term degree is applied to an equation only after it has been brought to the standard form (if it is not already in that form) of a polynomial equated to zero. Thus, if an equation is fractional in the unknown it has to be cleared of fractions with respect to that unknown, or if it is irrational in the unknown it has to be brought to a rational form, before we can speak of its degree. It goes without saying that often the reduction can be anticipated and the degree discovered.

As in the case of the quadratic equation, when an equation of any degree is given in the form,

$$\text{polynomial in } x=0$$

it will be seen that *to a root of the equation corresponds a factor of the polynomial.* This fact is so important that a formal proof will be given. The following lemma will first be proved:

Lemma. *Let $f(x)$ be any polynomial in x; when $f(x)$ is divided by $x-m$ the remainder will be $f(m)$, i.e., the result of substituting m for x in the polynomial.*

Suppose $f(x)$ divided by $x-m$, in the ordinary way, the division terminating when a remainder not involving x is reached. The quotient will be a polynomial, which denote by $q(x)$, and the remainder, which denote by r, will not involve x. Then from the meaning of division we have

$$f(x)=q(x).\ (x-m)+r$$

an equality which holds for all values of x. Put, then, $x=m$ and it follows that

$$\begin{aligned} f(m)&=q(m).\ (m-m)+r \\ &=r. \end{aligned}$$

Thus r, the remainder, is the result of substituting m for x in the polynomial.

Theorem. *Let $f(x)$ be any polynomial in x; then, if m is a root of the equation $f(x)=0$, must $x-m$ be a factor of $f(x)$.*

Since m is a root of $f(x)=0$, then $f(m)=0$. Let, now, $f(x)$ be divided by $x-m$; as has been seen, the remainder is $f(m)$. But $f(m)$ is known to be zero; therefore, the remainder is zero, and consequently $x-m$ is a factor of $f(x)$.

The proof of the converse theorem is immediate.

Ex. 1. Solve the equation

$$x^4-8x^2+15=0.$$

This equation is of degree 4 in the unknown x, but if x^2 is regarded as the unknown the equation is quadratic. Then

$$(x^2)^2-8(x^2)+15=0;$$

which is satisfied if $x^2=3$ or if $x^2=5$,
i.e., if $x=\pm\sqrt{3}$, or if $x=\pm\sqrt{5}$.

Thus, the given equation of the *fourth* degree has the *four* roots $+\sqrt{3}$, $-\sqrt{3}$, $+\sqrt{5}$, $-\sqrt{5}$, and the polynomial x^4-8x^2+15 expressed as a product of linear factors is

$$(x-\sqrt{3})(x+\sqrt{3})(x-\sqrt{5})(x+\sqrt{5}).$$

Ex. 2. Solve the equation

$$x^3-1=0.$$

Plainly must

$$(x-1)(x^2+x+1)=0.$$

The equation will be satisfied by those values of x which will make $x-1$ and x^2+x+1 separately zero, *i.e.*, by the value 1 and the roots of

$$x^2+x+1=0,$$

which are found to be

$$\frac{-1+\sqrt{-3}}{2} \text{ and } \frac{-1-\sqrt{-3}}{2}.$$

Thus, the given equation of the *third* degree has the *three* roots

$$1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}.$$

The equation may be put in the form $x^3=1$ whence it appears that x is the cube root of unity, so that there are three such cube roots, one real, the other two imaginary. The two imaginary roots are the roots of the equation,

$$x^2+x+1=0.$$

Denote these roots by m and n. Then

$$m^3=1,\ n^3=1,\ m+n=-1,\ mn=1.$$

Multiply both sides of the last relation by m^2. Then

$$m^3n=m^2.$$

But $m^3=1$.

$$\therefore \quad n=m^2,$$

and consequently, *each imaginary root is equal to the square of the other.* Therefore, if one imaginary root be denoted by ω, the three cube roots of unity are

$$1,\ \omega,\ \omega^2.$$

The relation $mn=1$ declares that *each imaginary root is the reciprocal of the other.*

These relations should be verified by actual computation.

Ex. 3. Solve

$$12x^4+4x^3-41x^2+4x+12=0.$$

This equation of the fourth degree is of special form, the co-efficients of the terms equidistant from the beginning and the end of the arranged polynomial being equal. The method of solution of such equations is as follows :

Divide each side of the equation by x^2; this is permissible since x is a root of the equation and it is seen that $x=0$ is not a root. Then

$$12x^2+4x-41+4.\frac{1}{x}+12.\frac{1}{x^2}=0$$

$$\therefore \quad 12\left(x^2+\frac{1}{x^2}\right)+4\left(x+\frac{1}{x}\right)-41=0.$$

Now $x^2+\frac{1}{x^2}$ differs from the square of $x+\frac{1}{x}$ by 2; supply then 2 within the brackets of the first term, *i.e.*, virtually 2×12, and correct by subtracting 24. Then

$$12\left(x+\frac{1}{x}\right)^2+4\left(x+\frac{1}{x}\right)-65=0.$$

If $x+\frac{1}{x}$ be regarded as the unknown this last is a quadratic equation. By solving it is seen to be satisfied by

$$x+\frac{1}{x}=\frac{13}{6}, \text{ or } x+\frac{1}{x}=-\frac{5}{2}.$$

Take (I) $$x+\frac{1}{x}=\frac{13}{6}$$

$$\therefore \quad 6x^2-13x+6=0, \text{ whence } x=\frac{2}{3} \text{ or } \frac{3}{2}.$$

Take (II) $$x+\frac{1}{x}=-\frac{5}{2}$$

$$\therefore \quad 2x^2+5x+2=0, \text{ whence } x=-2 \text{ or } -\frac{1}{2}.$$

Therefore the equation has the four roots

$$\tfrac{2}{3},\ \tfrac{3}{2},\ -2,\ -\tfrac{1}{2}.$$

The equations I and II both being of the form

$$x+\frac{1}{x}=k$$

or, when reduced, of the form

$$x^2-kx+1=0$$

will have the product of their roots equal to 1, *i.e.*, each the reciprocal of the other. On this account an equation of the type proposed will have roots which fall into pairs of reciprocal numbers. Such an equation is called a **reciprocal equation.**

Ex. 4. Solve.

$$(x+1)(x+2)(x+3)(x+4)=1680.$$

If the first and the last factors on the left are multiplied, and also the mean factors the equation takes the form

$$(x^2+5x+4)(x^2+5x+6)=1680.$$

Now, treat x^2+5x+4 as the unknown, for the time being, and denote it by u. Therefore, the equation becomes

$$u(u+2)=1680$$

or

$$u^2+2u-1680=0$$

$$\therefore \quad u=40 \text{ or } -42.$$

Take (I)

$$x^2+5x+4=40$$

$$\therefore \quad x^2+5x-36=0$$

$$\therefore \quad x=-9 \text{ or } +4.$$

Take (II)

$$x^2+5x+4=-42$$

$$\therefore \quad x^2+5x+46=0$$

$$\therefore \quad x=\frac{-5\pm\sqrt{-159}}{2}.$$

The roots of the proposed equation of the *fourth* degree — *four* in number — are

$$4, \ -9, \ \frac{-5+\sqrt{-159}}{2}, \ \frac{-5-\sqrt{-159}}{2},$$

of which two are imaginary.

Ex. 5. Solve.

$$2x^2+6x-\sqrt{x(x+3)-1}=17.$$

It is readily seen that if this equation is rationalized it will be of the fourth degree. To avoid passing to an equation of this degree it is well to examine the forms appearing in the equation. By writing the equation thus:

$$2(x^2+3x)-\sqrt{x^2+3x-1}-17=0$$

it is seen that x occurs only in the grouping x^2+3x, or x^2+3x-1. Since the radical sign requires that x^2+3x-1 be treated as one quantity, we modify the earlier grouping and write the equation thus:

$$2(x^2+3x-1)-\sqrt{x^2+3x-1}-15=0.$$

Denote $\sqrt{x^2+3x-1}$ by u.

$$\therefore\ 2u^2-u-15=0.$$

which is a quadratic in u.

$$\therefore\ u=3 \text{ or } -\tfrac{5}{2}.$$

Take (I) $u=3$.

$$\therefore\ \sqrt{x^2+3x-1}=3$$
$$\therefore\ x^2+3x-1=9$$
$$\therefore\ x^2+3x-10=0$$
$$\therefore\ x=2 \text{ or } -5.$$

Take (II) $u=-\frac{5}{2}$.

$$\therefore\ \sqrt{x^2+3x-1}=-\tfrac{5}{2}$$
$$\therefore\ x^2+3x-1=\tfrac{25}{4}$$
$$\therefore\ x^2+3x-\tfrac{29}{4}=0$$
$$\therefore\ x=\frac{-3\pm\sqrt{38}}{2}.$$

Thus, the original equation, virtually of the *fourth* degree, has the *four* roots

$$2,\ -5,\ \frac{-3+\sqrt{38}}{2},\ \frac{-3-\sqrt{38}}{2}$$

Of these roots, only the first two satisfy the equation if in evaluating $\sqrt{x^2+3x-1}$ we take only the positive root. The solution does not regard this limitation, and in its larger sense the equation should be said to have four roots.

EXERCISES

Solve the following equations :

1. $10x^4-29x^2+21=0.$
2. $35x^4-39x^3-4x^2-39x+35=0.$
3. $(x+3)(x+4)(x+5)(x+6)=5040.$
4. $(2x+1)(x-3) - \sqrt{x^2+x(x-5)+11}=16.$
5. $\dfrac{3x}{4+x^2}+\dfrac{4+x^2}{3x}=\dfrac{25}{12}.$
6. $12x^4-8x^3-15x^2-8x+12=0.$
7. $x^4-1=0.$
8. $(x^2+7x-5)(x+3)(x+4)=1050.$
9. $\sqrt[3]{x+20}-\sqrt[3]{x+1}=1.$
10. $(x+3)^2-\sqrt{(x+1)(x+5)+25}=89.$

7. Simultaneous Equations of the First Degree. In the expression $2x+3y-13$, the numbers 2, 3, -13 are constants and the numbers x, y are variables, and the expression is a function of x and y depending for its value on the values assigned to x and y.

Consider now the equation

$$2x+3y-13=0$$

or which is the same thing

$$2x+3y=13.$$

Plainly, it is an easy matter to find solutions of this equation, for if to y be assigned any value whatever, the equation determines a value of x which with that of y constitutes a solution. Thus, if $y=1$ it is found that $x=5$, and thus $x=5$, $y=1$ is a solution. The equation then imposes only a partial restraint upon the variables in the expression $2x+3y-13$, making the variation of one determine the variation in the other. The following table exhibits a series of solutions:

$x=6\frac{1}{2}$	5	$3\frac{1}{2}$	2	$\frac{1}{2}$	-1	.....
$y=0$	1	2	3	4	5	.....

Similar remarks apply to the equation

$$3x+2y-12=0$$

or

$$3x+2y=12,$$

and a series of solutions is given in the table,

$x=4$	$3\frac{1}{3}$	$2\frac{2}{3}$	2	$1\frac{1}{3}$	$\frac{2}{3}$	.....
$y=0$	1	2	3	4	5	.....

Among the solutions of the two equations we notice one, namely $x=2$, $y=3$, which is common, and we inquire whether it is the only common solution. In other words, we ask for all values of x and y that will satisfy the two equations

$$2x+3y=13 \qquad (1)$$
$$3x+2y=12 \qquad (2)$$

at the same time. Since we now suppose x and y to denote any such values, the x and the y in the two equations are the same. Multiplying both sides of (1) by 2 and both sides of (2) by 3 we have the equivalent set

$$4x + 6y = 26 \qquad (3)$$
$$9x + 6y = 36. \qquad (4)$$

Therefore, by subtraction

$$5x = 10$$

or

$$x = 2.$$

Substituting this value of x in either equation we find the value of y to be 3, so that $x = 2$, $y = 3$ is the only solution common to the two equations. Thus, *the two linear equations in x and y serve completely to determine the values of x and y.*

In the same way the two general linear equations in two unknowns

$$ax + by = c,$$
$$a'x + b'y = c',$$

are seen to have one and only one solution, namely,

$$x = \frac{cb' - c'b}{ab' - a'b}, \quad y = \frac{ca' - c'a}{ba' - b'a}.$$

If it is a question of linear equations in three variables, a similar examination will shew that:

(1) *One linear equation allows any values whatever to be assigned to two of the variables, these two values determining that of the remaining variable.*

(2) *Two linear equations allow any value whatever to be assigned to one of the variables, this value determining those of the remaining variables.*

(3) *Three linear equations determine completely the values of the three variables.*

These statements are made on the supposition that the equations are *independent* and *consistent.* Thus, the equations

$$x + 2y - z = 3$$
$$2x + 4y - 2z = 6$$

are not independent, the one following from the other, and should not be spoken of as two equations; also, the equations

$$x + 2y - z = 3$$
$$x + 2y - z = 7$$

cannot both be true, *i.e.*, are inconsistent, x, y, z, being supposed finite, and, therefore, cannot both aid in determining values of x, y, z.

EXERCISES

Solve the following systems of equations:

1. $5x+7y=50,$
 $4x+5y=37.$

2. $\frac{x}{3}-\frac{y}{6}=\frac{1}{2},$
 $\frac{x}{5}-\frac{3y}{10}=\frac{1}{2}.$

3. $\frac{x+y}{2}-\frac{x-y}{3}=16,$
 $\frac{x+y}{3}+\frac{x-y}{4}=22.$

4. $x+2y+3z=14,$
 $2x+3y+5z=23,$
 $3x+5y+7z=34.$

5. $x-y+z=11,$
 $2x-3y+9z=21,$
 $3x+4y-5z=17.$

8. **Simultaneous Equations Involving the Unknowns to a Degree Higher than the First.** In this Art. will be given the solutions of certain examples which illustrate classes of equations which may be solved without appeal to the solutions of equations of the third or the fourth degree.

Ex. 1. Solve,

$$\left.\begin{array}{r} 2x-3y=3 \\ x^2-xy+2y^2+x-4y=7 \end{array}\right\}$$

Here one of the given equations is linear and the other quadratic in the unknowns. Substitute in the quadratic for one of the unknowns its value, found from the linear equation, in terms of the other unknown.

From the linear equation

$$x=\frac{3y+3}{2}.$$

Therefore, by substitution in the quadratic equation,

$$\left(\frac{3y+3}{2}\right)^2-\left(\frac{3y+3}{2}\right)y+2y^2+\frac{3y+3}{2}-4y=7,$$

or

$$11y^2+2y-13=0,$$

a quadratic in y, as was easily foreseen.

$$\therefore\ y=1 \text{ or } -\tfrac{13}{11}.$$

If $y=1$, x, which equals $\frac{3y+3}{2}$, is found to be 3.

If $y=-\frac{13}{11}$, x which equals $\frac{3y+3}{2}$, is found to be $-\frac{3}{11}$.

The solutions then are

$$(x=3,\ y=1),\ (x=-\tfrac{3}{11},\ y=-\tfrac{13}{11}).$$

Any system of equations of the kind in question will, then, admit two solutions.

Ex. 2. Solve

$$\left.\begin{aligned}2x^2-4xy+3y^2&=6\\ x^2+\ xy-2y^2&=7\end{aligned}\right\}\quad(1)$$

The two equations are here quadratics. In general, two such equations can be solved only by solving a general equation of the fourth degree. The equations given are, however, of special form in that they contain no terms of one dimension.

Since the equations are presumably satisfied by some values of x and y, the value of y will be equal to the value of x multiplied by some number, not yet known. We may put, then,

$$y=mx\quad(2)$$

where m is a new unknown.

Substitute in the given equations: then

$$\left.\begin{aligned}2x^2-4mx^2+3m^2x^2&=6\\ x^2+mx-2m^2x^2&=7\end{aligned}\right\}\quad(3)$$

or

$$\left.\begin{aligned}x^2(2-4m+3m^2)&=6\\ x^2(1+m-2m^2)&=7\end{aligned}\right\}\quad(4).$$

Then, by the division of equal quantities, since x plainly cannot be zero,

$$\frac{2-4m+3m^2}{1+m-2m^2}=\frac{6}{7}.$$

Then, clearing of fractions, we have

$$33m^2-34m+8=0$$
$$\therefore \quad m=\tfrac{2}{3} \text{ or } \tfrac{4}{11}.$$

Take first $m=\frac{2}{3}$. Then substituting in the last of equations (4) we have

$$x^2(1+\tfrac{2}{3}-\tfrac{8}{9})=7$$
$$\text{or} \quad x=+3 \text{ or } -3$$

$$\left.\begin{array}{l} \text{If } x=+3,\ y=\ \ 2 \\ \text{If } x=-3,\ y=-2 \end{array}\right\}, \text{ since } y=mx, \text{ and } m=\tfrac{2}{3}.$$

Take next, $m=\frac{4}{11}$. Then, as before,

$$x^2(1+\tfrac{4}{11}-\tfrac{32}{121})=7$$
$$\text{or} \quad x=+\frac{11}{\sqrt{19}} \text{ or } -\frac{11}{\sqrt{19}}.$$

$$\left.\begin{array}{l} \text{If } x=+\dfrac{11}{\sqrt{19}},\ y=\dfrac{4}{\sqrt{19}} \\ \text{If } x=-\dfrac{11}{\sqrt{19}},\ y=-\dfrac{4}{\sqrt{19}} \end{array}\right\}, \text{ since } y=mx, \text{ and } m=\tfrac{4}{11}.$$

Thus, the *two quadratic equations in two unknowns yield the four solutions*

$x=+3$	-3	$+\dfrac{11}{\sqrt{19}}$	$-\dfrac{11}{\sqrt{19}}$
$y=+2$	-2	$+\dfrac{4}{\sqrt{19}}$	$-\dfrac{4}{\sqrt{19}}$

Ex. 3. Solve

$$\left.\begin{array}{l} x^3-y^3=98 \\ x\ -y\ =\ 2 \end{array}\right\}.$$

We have here a system of one linear and one cubic equation. Since $x-y$ is a factor of x^3-y^3, and is not equal to zero, we have by division by equal values,

$$x^2+xy+y^2=49.$$

Substituting for x from $x-y=2$, we have

$$(2+y)^2+(2+y)y+y^2=49$$
$$\therefore \quad y^2+2y-15=0$$
$$\therefore \quad y=3, \text{ or } -5.$$
$$\text{If } y=3, \text{ then } x=5, \text{ since } x-y=2.$$
$$\text{If } y=5, \text{ then } x=-3, \text{ since } x-y=2.$$

Therefore, two solutions are yielded, namely,

$$(x=5,\ y=3),\ (x=-3,\ y=-5).$$

These solutions would also have been obtained if we had substituted for x, from $x-y=2$, in the original cubic.

EXERCISES

Solve the following systems of equations:

1. $x+y=7,$
$x^2+3xy-5y^2+2x-y=12.$

2. $x+y=8,$
$x^2+y^2=34.$

3. $x^2+xy=30,$
$6x^2-y^2=5.$

4. $2x^2-3xy+5y^2=7,$
$x^2+5xy-11y^2=3.$

5. $x^3+y^3=189,$
$x+y=9.$

6. $x^4+x^2y^2+y^4=21,$
$x^2+xy+y^2=7.$

7. $3x+4y=18,$
$2x^2-7xy-y^2+5x+11y=0.$

8. $xy=42,$
$x^2+y^2=85.$

9. $4x+6y=xy,$
$12x+9y=2xy.$

10. $2(x^2+y^2)-3(x+y)=5,$
$4xy=15.$

11. $xy(x+y)=30,$
$xy+x+y=11.$

12. $x^2+2xy=16,$
$xy+y^2=15.$

13. $x^2-2y^2=4y,$
$3x^2+xy-2y^2=16y.$

14. $x-y=4,$
$x^3-y^3=316.$

15. $xy(x+y)=84,$
$\frac{1}{x}+\frac{1}{y}=\frac{7}{12}.$

(Note.—Denote the function by y in place of u as earlier. The axes will be X′OX, YOY′, the former the x-axis and the latter the y-axis for measurements determining the curve.)

EXAMPLES

I

1. Denote the function $2x+3$ by y (in place of u as earlier, taking X′OX, Y′OY as axes, the former the x-axis, the latter the y-axis), take x_1 and x_2 any two numbers as giving certain lengths, find the values of y for these values of x, and, denoting them by y_1 and y_2 mark relatively to the axes the corresponding points P_1 and P_2.

Shew that $\dfrac{y_2-y_1}{x_2-x_1}=2$, which implies that P_2 lies relatively to P_1 on a *slope* or *gradient* of 2.

Hence shew that *all* points of the graph of $2x+3$ are on a straight line, and in the graph point out the significance of the constants 2 and 3 occurring in $2x+3$.

2. Treat as in the preceding example the functions

$\frac{2}{3}x-\frac{1}{4}$, $5-$ [illegible] x, $ax+b$.

3. Noting that the equation

$$2x+3y=24$$

gives y as a function of x, namely

$$y=\frac{24-2x}{3},$$

construct the graph of the function.

(NOTE.—This graph, seen to be a straight line, is also spoken of as the graph of the *equation* $2x+3y=24$).

4. It is known that twice A's money exceeds three times B's money by 1 (dollar). Shew that this does not determine the amounts held by either, but that if x and y denote these amounts, then x and y correspond to some point on a certain graph,—a straight line.

5. It is known that twice A's money exceeds three times B's money by 1 (dollar), and also that the sum of A's money and B's money is 8 (dollars). If x and y denote the amounts held by A and B, shew that x and y correspond to a point on each of two graphs and hence find x and y.

6. Solve algebraically and graphically each of the following systems of equations:

(i) $3x-4y=2,$ $\quad$ $4x-3y=12.$

(ii) $3x+4y=25.$ $\quad$ $3x-2y=1.$

7. Shew that the system of equations

$$2x-3y=4,\ 4x-6y=7$$

does not admit solution.

Represent the system graphically.

II

1. If x denotes the temperature Centigrade, in degrees, and y the same temperature Fahrenheit, shew that

$$9x-5y+160=0.$$

Construct a graph that will give the Fahrenheit reading corresponding to any given Centigrade reading.

2. A number is expressed by two digits. The sum of the digits is 11, and the tens' digit exceeds the units' digit by 2. Write down the equations which give the number and solve them graphically.

From the graphs pick out all the numbers that satisfy each condition.

3. On the graph paper draw any straight line, and, the axis having been marked, employ any suitable measurements to write down the equation of which the line is the graph.

Comment on the absolute accuracy of the result.

4. Solve graphically the system

$$5x-3y=9,\ 2x+3y=12.$$

5. Solve graphically the system

$$4x-3y=11,\ 2x+4y=27.$$

6. Shew that there is no set of values of x and y that will satisfy the three equations

$$3x-2y=5,\ 5x-3y=9,\ 2x+3y=20.$$

Illustrate graphically.

7. Shew that there is a set of values of x and y that will satisfy the three equations

$$x+3y=9,\ 3x-y=7,\ 5x+y=17.$$

III

1. Shew that for *real* values of x, the minimum value of $2x^2-3x-5$ is $-6\frac{1}{8}$, that $x=\frac{3}{4}$ gives this value, and that, k being any number, $\frac{3}{4}+k$ and $\frac{3}{4}-k$ give the function the same value. Employ these facts to construct a graphical representation of the function.

2. Shew that for real values of x, the maximum value of $4-x-3x^2$ is $3\frac{3}{4}$, that $x=-\frac{1}{6}$ gives this value, and that k being any number $-\frac{1}{6}+k$ and $-\frac{1}{6}-k$ give the function the same value. Employ these facts to construct a graphical representation of the functions.

3. Shew that the minimum value of $2x^2-3x+5$ is $3\frac{7}{8}$, and therefore that $2x^2-3x+5$ cannot have the value zero.

Explain how it is, then, that we can solve the equation

$$2x^2-3x+5=0.$$

4. Resolve into factors linear in x

$$2x^2-x-3,\ 2x^2-x-2,\ 2x^2-x+1$$

pointing out any differences in the nature of the results.

5. Construct the graphs of the following equations:

(i) $y=x^2-5x+6$, (ii) $y=6-x-x^2$, (iii) $x+y-x^2=0$.

6. Represent graphically, referring each to the one set of axes, the functions

$$x^2-3x+2, \text{ and } 3x+5$$

and from the graphs find the solution of the system of equations

$$y=x^2-3x+2, \text{ and } y=3x+5.$$

Test the result by solving this system.

7. Represent graphically the function $2x^2-3x-5$, and from the graph estimate the roots of the equation $2x^2-3x-5=3$.

Test results by algebraic work.

IV

1. Construct with great care the graph of the function x^2, and employ it to obtain approximations to $\sqrt{0 \cdot 5}$, $\sqrt{1 \cdot 5}$, $\sqrt{2}$, $\sqrt{2 \cdot 75}$, $\sqrt{3}$.

2. Solve algebraically and graphically the system

$$2x + 3y = 13, \;\; x^2 - 5x + 3y = 9.$$

3. Examine the relations that must exist between the graphs of the two functions $x^2 - 2x - 8$ and $8 + 2x - x^2$.

4. Find the minimum value of each of the functions

$$x^2 - 4x + 7, \;\; 2x^2 + 3x - 7, \;\; x^2 + px + q,$$

stating the argument in each case.

5. Find the maximum value of each of the functions

$$2 - 5x - x^2, \;\; 7 - 4x - 3x^2, \;\; q + px = x^2,$$

stating the argument in each case.

6. Shew that the value of x which gives the function $x^2 + px + q$ its minimum value is one-half the sum of the roots of the equation $x^2 + px + q = 0$.

7. Shew that the value of x which gives the function $q + px - x^2$ its maximum value is one-half the sum of the roots of the equation $q + px - x^2 = 0$.

V

1. Divide the straight line (1) of length 12, (2) of length a, into two parts so that the rectangle contained by the parts may be the greatest possible.

2. A square is inscribed to a square of side 12. Find when the inscribed square has the smallest area.

Shew that this may be deduced from the preceding example.

3. The base of a semi-circle whose radius is a, is divided into two parts, and on these as bases semi-circles are described. How must the base be divided if the area bounded by the three arcs is a maximum?

4. Divide a straight line of length 40 inches into two parts such that the hypothenuse of the right-angled triangle of which the two parts are the sides shall be the least possible.

5. An article is sold at a loss of as much per cent. as it cost in dollars. Shew that it cannot be sold for more than 25 dollars.

6. ABC is an equilateral triangle (1) of side 2, (2) of side a, and P the middle point of BC. Of all the isosceles triangles PQR, where Q and R are on the sides AB and AC of the given triangle, find the one of greatest area.

7. ABCD is a square in which is inscribed a square PQRS. Find the minimum size of PQRS.

VI

1. If m and n are the roots of the equation $ax^2+bx+c=o$, express in terms of the known numbers a, b, c the following:

$$m^2+mn+n^2,\ m^3+n^3,\ \frac{1}{m^2n}+\frac{1}{mn^2},\ \frac{m^2}{n}+\frac{n^2}{m}.$$

2. One root of a quadratic equation with rational coefficients is $3+\sqrt{7}$: shew what the other root must be, and find the equation.

3. In a certain quadratic equation the coefficient of x^2 is the same as the absolute term. One root is known to be $\frac{5}{7}$; find the other root, and the equation.

4. One root of a quadratic equation with real coefficients is $5+\sqrt{-3}$. Find the other root, and the equation.

5. The equations $10x^2-3x-18=0$ and $6x^2-17x+12=0$ are known to have a common root. From this fact alone, *i.e.*, without actually solving the equations, find the common root, and deduce the remaining root in each equation.

6. Find the relation that must exist among the coefficients of the equations $ax^2+bx+c=0$ and $px^2+qx+r=0$ in order that they may have a common root.

Shew that the coefficients of the equations in the preceding example satisfy this relation.

7. In the equation $ax^2+bx+c=0$, shew that, if a and c are the one negative and the other positive, the roots are real, one being positive the other negative.

Shew also how to determine which of these roots is numerically the greater.

Examine in this regard the equation $2x^2-7x-5=0$.

VII

1. Find the values of m if the equation

$$(m-1)x^2-(4m+4)x+(7m+1)=0$$

is known to have equal roots. Further, take each of the values of m found and find those equal roots.

2. One root of the equation

$$x^2-(k+1)x+(2k+1)=0$$

is known to exceed the other by 2. Find the value of k and find the roots that meet this condition.

3. Given that the difference between the roots of the equation $ax^2+bx+c=0$ is equal to the difference between the roots of the equation $px^2+qx+r=0$, shew that $p^2(b^2-4ac)=a^2(q^2-4pr)$.

4. If $x^2=25$, explain on what grounds it is stated that $x=+5$ or -5, implying that there is no other value of x that will satisfy the equation.

5. Write down the equation whose only roots are 3, 5, and 7, justifying the answer.

6. Given that $z=\frac{5}{7}$ is one root of the equation

$$77z^3+15z^2-43z-5=0,$$

find the other roots.

7. Construct the equation whose roots exceed twice those of the equation $3x^2-5x-11=0$, by 7.

VIII

1. Plot the curve (*i.e.*, construct the graph) of the function

$$x^3-2x^2-5x+5,$$

in other words, represent graphically the equation

$$y=x^3-2x^2-5x+5$$

for values of x between -3 and 4, and state any inferences as to the nature and values of the roots of the equation

$$x^3-2x^2-5x+5=0.$$

2. Plot the curve of the function

$$x^3 - 5x^2 + 11x - 14$$

between $x = 0$ and $x = 4$, and obtain an approximate value of a root of the equation

$$x^3 - 5x^2 + 11x - 14 = 0.$$

3. Construct the graph of the equation $y = x^3$, for values of x between -3 and $+3$, and employ it to get an approximate value of $\sqrt[3]{0{\cdot}5}$, $\sqrt[3]{0{\cdot}7}$, $\sqrt[3]{1{\cdot}5}$, $\sqrt[3]{2}$, $\sqrt[3]{2{\cdot}5}$.

4. The volume of a gas under certain pressures is given by the following table, the pressure being in pounds to the square inch, and the volume in cubic feet:

Pressure	12	15	18	24	27
Volume	90	72	60	45	40

Plot a curve showing the variation in volume under the changing pressure and conjecture the volume under pressure 21.

If v measures the volume, and p the pressure, what numerical relation would seem to connect v and p?

5. The population of a city at certain dates is given as follows:—1850; 37,000: 1860; 43,600: 1870; 52,700: 1880; 61,400: 1890; 66,300: 1900; 69,800: 1910; 75,500.

Represent this table graphically.

6. The quoted prices of a certain stock at intervals of 15 days were:—124, 119, 111, 100, 96, 94.

Represent this graphically.

7. The temperatures throughout a day at intervals of two hours were recorded as follows, starting at midnight:—38°, 36°, 30°, 31°, 35°, 42°, 48°, 55°, 60°, 54°, 50°, 45°.

Represent the changing temperature graphically.

IX

1. Solve

$$(x+3)(x+5)(x+7)(x+9)=3465.$$

2. Solve

$$2x^2+6x-41=\sqrt{x^2+3x+7}.$$

3. Solve

$$9x^4-51x^3+88x^2-51x+9=0.$$

4. Solve

$$x^5-1=0.$$

5. Solve

$$x^6-1=0.$$

6. Solve

$$\sqrt{2x+3}+\sqrt{3x+3}=11.$$

7. Solve

$$\sqrt{1+x^2}-\sqrt{1-x^2}=\sqrt{1-x^4}.$$

X

1. Solve

$$\begin{aligned} 3x^2-5xy-7x+11y+14&=0,\\ 4x-y-7&=0.\end{aligned}$$

2. Solve

$$\begin{aligned} xy+x+y&=11,\\ xy(x+y)&=30.\end{aligned}$$

3. Solve

$$\begin{aligned} x^2+2xy&=16,\\ xy+y^2&=15,\end{aligned}$$

4. Solve

$$\begin{aligned} x^2-2y^2&=4y,\\ 3x^2+xy-2y^2&=16y.\end{aligned}$$

5. Solve

$$\begin{aligned} x+y&=4,\\ x^4+y^4&=82.\end{aligned}$$

6. Solve

$$\begin{aligned} 3x+5y&=4x^2-6y^2,\\ 2x+12y&=x^2+y^2.\end{aligned}$$

7. Solve

$$\frac{x}{y}+\frac{y}{x}=\frac{74}{35},$$

$$\frac{1}{x}+\frac{1}{y}=\frac{12}{35}.$$

XI

1. Employ the Lemma of page 17 to shew that

(i) $x^n - a^n$ is divisible by $x - a$;

(ii) $x^n + a^n$ is divisible by $x + a$ if n is an odd integer.

2. Shew that a polynomial in x is divisible by $x - 1$ if the sum of its coefficients is equal to zero.

Employ this fact to solve the equation

$$3x^3 - 5x^2 + 11x - 9 = 0.$$

3. Shew that a polynomial in x is divisible by $x + 1$ if the sum of the coefficients of the odd powers of x is equal to the sum of the coefficients of the even powers.

Write down a polynomial of degree four that is divisible by $x + 1$.

4. If $x^2 + px + 21$ is divisible by $x - 3$, find the value of p.

5. Solve the equation

$$x^4 + 2x^3 - 6x^2 - 2x + 5 = 0.$$

6. Solve

$$x^5 + 1 = 0.$$

7. If $x^2 - 8x + p$ is divisible by $x - 5$, find the value of p.

CHAPTER II

RATIO AND PROPORTION

1. **Preliminary.** When two different numbers, as for example 2 and 3, are given, the fact of difference may be regarded in different ways. Thus, we may say that the absolute difference is 1, meaning that 2 is less than 3 by 1, or that 3 is greater than 2 by 1; or we may consider the relative values of the two numbers and say that 2 is two thirds of 3 or that 3 is three halves of 2. In regarding two numbers in this latter way we arrive at the concept of **ratio,** and we see that the ratio of 2 to 3 (in symbols $2:3$) is expressed by the fraction $\frac{2}{3}$.

The ratio of two numbers being expressed by a fraction, we may speak of the value of a ratio and may study the properties of ratios in the fractions by which they are represented. Indeed, all the theorems proved with respect to fractions are theorems in ratios.

In the ratio $2:3$, the numbers 2 and 3 are called the terms of the ratio, the former the antecedent, and the latter the consequent.

If two numbers are equal we still speak of their ratio; this ratio is expressed by the number 1 and is called a **ratio of equality.** A ratio of two positive numbers in which the antecedent exceeds the consequent is called a **ratio of greater inequality;** one, in which the antecedent is less than the consequent, a **ratio of less inequality.**

One ratio is equal to, greater than, or less than another according as the fraction which represents that ratio is equal to, greater than, or less than the fraction representing the other ratio. When two ratios are equal the four numbers in order are said to be in **proportion.** Thus, 2, 3, 10, 15 are in proportion since

$$\frac{2}{3}=\frac{10}{15}.$$

The statement of the proportion in symbols is

$$2:3::10:15$$

which is read, *two is to three as ten is to fifteen.* To find whether or not four given numbers are in proportion it is necessary only to examine whether or not two fractions are equal.*

When several numbers in sequence as $a, b, c, d, \ldots\ldots$ are such that $a:b::b:c::c:d::\ldots\ldots$, these numbers are said to be in **continued proportion.**

If three numbers $a, b, c,$ are in continued proportion b is said to be a **mean proportional** between a and c, and c is said to be a **third proportional** to a and b.

If there are several (say three) ratios as $a:b$, $c:d$, $e:f$, the ratio $ace:bdf$ is said to be **compounded** of the given ratios, and hence if there are several (say four) numbers in sequence p, q, r, s, then the ratio $p:s$ is said to be compounded of the ratios $p:q$, $q:r$, $r:s$.

The ratios $a^2:b^2$, $a^3:b^3$, $\ldots\ldots$ are called the **duplicate**, the **triplicate** $\ldots\ldots$ of the ratio $a:b$.

2. The following propositions are of importance:

(I). *A ratio of less inequality is increased, a ratio of greater inequality is diminished, and a ratio of equality is unchanged by the addition of the same positive number to each term.*

Let the ratio be $a:b$ where a and b are positive, and let the positive number k be added to each term so that the resulting ratio is $a+k:b+k$. The two ratios are expressed by the fractions

$$\frac{a}{b},\ \frac{a+k}{b+k}.$$

Then

$$\frac{a+k}{b+k}-\frac{a}{b}=\frac{b(a+k)-a(b+k)}{b(b+k)}$$

$$=\frac{(b-a)k}{b(b+k)}.$$

Now, k, b, $b+k$ are positive so that the sign of this last fraction is that of $b-a$. Therefore the difference $\frac{a+k}{b+k}-\frac{a}{b}$ is positive, negative or

* In contrast with the definition and treatment of proportion here given, Euclid's definition and treatment appear difficult and forced. This is due to the fact that Euclid provides for cases not here contemplated, namely, those in which appear irrational numbers *of whatever kind*. To establish a theory of operations of such numbers is certainly not easier than to comprehend the meaning and feel the beauty of his definition.

zero according as b is greater than, less than, or equal to, a. Thus, the ratio $a:b$ has been increased if b is greater than a, *i.e.*, if the ratio is one of less inequality; diminished, if b is less than a, *i.e.*, if the ratio is one of greater inequality; and unchanged if b is equal to a, *i.e.*, if the ratio is one of equality.

(II). *If $a:b$ and $c:d$ are two unequal ratios of positive quantities, the ratio $a+c:b+d$ is intermediate in value to these two ratios.*

The given ratios are expressed by

$$\frac{a}{b}, \frac{c}{d}.$$

Let $\frac{a}{b}$ be the greater of these fractions and denote its value by k. Then

$$\frac{a}{b}=k$$

$$\therefore \quad a=bk$$

Also $$\frac{c}{d}<k$$

$$\therefore \quad c<dk$$

$$\therefore \quad a+c<bk+dk,$$

$$i.e., \quad a+c<(b+d)k$$

$$\therefore \quad \frac{a+c}{b+d}<k; \ i.e., \ \frac{a+c}{b+d}<\frac{a}{b}.$$

In like manner $$\frac{a+c}{b+d}>\frac{c}{d}.$$

Therefore, the ratio $a+c:b+d$ lies in value between $a:b$ and $c:d$.

This theorem can easily be generalized.

(III). *If four quantities are proportionals, the product of the extremes is equal to the product of the means.*

Let $a:b::c:d$; it is required to prove that $ad=bc$. It is given that

$$\frac{a}{b}=\frac{c}{d}.$$

Multiply each side of this equality by bd. Then

$$ad=bc.$$

Conversely if $ad=bc$ then shall $a:b::c:d$.

For if

$$ad = bc$$

then, by dividing each side by bd, we obtain

$$\frac{a}{b} = \frac{c}{d},$$

i.e., $$a : b :: c : d.$$

Cor. If $ad = bc$, it is plain also that

$$\frac{a}{c} = \frac{b}{d};$$

therefore, if $a : b :: c : d$ it follows that $a : c :: b : d$.

(IV). *If $a : b :: c : d$ then each ratio is equal to the ratio expressed by*

$$\frac{la + mc}{lb + md}.$$

Since $\frac{a}{b}$ and $\frac{c}{d}$ are equal, we may denote their common value by v.

Then $a = bv$, $c = dv$, and

$$\frac{la + mc}{lb + md} = \frac{lbv + mdv}{lb + md}$$

$$= \frac{v\,(lb + md)}{lb + md}$$

$$= v, \text{ the common value of } \frac{a}{b}, \frac{c}{d}.$$

The theorem may be shewn to be true without introducing the quantity v.

For if

$$\frac{a}{b} = \frac{c}{d}$$

then $$\frac{a}{c} = \frac{b}{d}$$

$$\therefore \frac{la}{c} = \frac{lb}{d}$$

$$\therefore \frac{la}{c} + m = \frac{lb}{d} + m$$

i.e., $$\frac{la + mc}{c} = \frac{lb + md}{d}$$

$$\therefore \frac{la + mc}{lb + md} = \frac{c}{d}\left(= \frac{a}{b}\right).$$

The theorem may be stated thus:

If two fractions are equal, each is equal to the quotient of the sum (or difference) of any multiples of their numerators by the sum (or difference) of the same multiples of corresponding denominators.

(V). *If* $\frac{a}{b}=\frac{c}{d}$, *then* $\frac{la+mb}{pa+qb}=\frac{lc+md}{pc+qd}$,

Since $\frac{a}{b}=\frac{c}{d}$ it follows that $\frac{a}{c}=\frac{b}{d}$. By theorem (IV) each of these latter is equal to $\frac{la+mb}{lc+md}$, as also to $\frac{pa+qb}{pc+qd}$.

$$\therefore \frac{la+mb}{lc+md}=\frac{pa+qb}{pc+qd}$$

$$\therefore \frac{la+mb}{pa+qb}=\frac{lc+md}{pc+qd}.$$

The theorem may be given verbal statement.

3. **Illustrative Examples.** The following examples may repay study:

Ex. 1. If $ax-by=0$, find the ratio of x to y.

Since

$$ax-by=0,$$

it follows that

$$ax=by.$$

Therefore, dividing each side by ay, we have

$$\frac{x}{y}=\frac{b}{a}$$

i.e., $\quad x:y::b:a.$

Thus: A homogeneous linear equation in x and y, *i.e.*, one which, when brought to the form in which zero is the right number, has every term of one dimension in x and y, while allowing x and y each to have different values, determines the ratio of x to y.

Ex. 2. If

$$ax+by+cz=0$$

and

$$px+qy+rz=0$$

find the ratios $x:y:z$.

Here we have two homogeneous linear equations in x, y, z. As has been seen two such equations do not determine x, y, z, and x, y, z may vary while satisfying the equations.

Multiplying each member of the first equation by r and each member of the second by c, we have,

$$arx+bry+crz=0$$
$$cpx+cqy+crz=0$$

Therefore, eliminating z, we have

$$x(cp-ar)-y(br-cq)=0$$

$$\therefore \quad x(cp-ar)=y(br-cq)$$

$$\therefore \quad \frac{x}{y}=\frac{br-cq}{cp-ar}$$

or $$\frac{x}{br-cq}=\frac{y}{cp-ar}.$$

By eliminating y we have in like manner

$$\frac{x}{br-cq}=\frac{z}{aq-bp}.$$

Therefore, the relations

$$\frac{x}{br-cq}=\frac{y}{cp-ar}=\frac{z}{aq-bp}$$

give the ratios $x:y:z$.

Thus, while x, y, z may vary, the variation is such as to keep the *ratios* $x:y:z$ constant.

The expressions for these ratios are important, and it is easy to devise a method of writing them down from the three triads

$$\left\{\begin{matrix} x & y & z \\ a & b & c \\ p & q & r \end{matrix}\right\}.$$

Ex. 3. Solve

$$2x-3y+z=0,$$
$$x+2y-2z=0,$$
$$x^2+2y^2-z^2=68.$$

From the first two equations

$$\frac{x}{6-2}=\frac{y}{1+4}=\frac{z}{4+3}$$

$$i.e., \quad \frac{x}{4}=\frac{y}{5}=\frac{z}{7}$$

Denote the common value of these ratios by k.

$$\therefore \quad x=4k,\ y=5k,\ z=7k.$$

Substitute in the third of the given equations.

$$\therefore\ 16k^2+50k^2-49k^2=68$$
$$\therefore\ k^2=4$$
$$\therefore\ k=+2 \text{ or } -2$$

$\therefore$ the solutions are

(1) $(x=8,\ y=10,\ z=14)$,
(2) $(x=-8,\ y=-10,\ z=-14)$.

EXAMPLES

1. If $12x^2-41xy+35y^2=0$ find the ratio of x to y.

2. For what value of x will $5+x$, $7+x$, $11+x$ be in continued proportion?

3. Find the number which added to each term of the ratio $5:8$ will yield the ratio $4:5$.

4. The lengths of two rectangles are in the ratio $a:a'$; their breadths are in the ratio $b:b'$. Find the ratio of their areas.

5. Find the mean proportional between 12 and 75.

6. If $\frac{a}{b}=\frac{c}{d}$, then will

(1) $$\frac{a^2b+ab^2}{a^3-b^3}=\frac{c^2d+cd^2}{c^3-d^3};$$

(2) $$\frac{3a^{10}+7b^{10}}{a^5b^5}=\frac{3c^{10}+7d^{10}}{c^5d^5}.$$

Shew that these are illustrations of the theorem: *If* $\frac{a}{b}=\frac{c}{d}$ *then any fraction formed with numerator and denominator homogeneous in a and b will be equal to the fraction similarly formed of c and d.*

7. If $\frac{a}{b}=\frac{c}{d}$, then

(1) $$\frac{ma^2+nc^2}{mb^2+nd^2}=\frac{ac}{bd};$$

(2) $$\frac{3a^3+5c^3}{3b^3+5d^3}=\frac{4a^2c+7ac^2}{4b^2d+7bd^2}.$$

8. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then

(1) $$\frac{la^2+mc^2+ne^2}{lb^2+md^2+nf^2}=\frac{pce+qea+rac}{pdf+qfb+rbd};$$

(2) $$\frac{a^3+c^3+e^3}{b^3+d^3+f^3}=\frac{ace}{bdf};$$

(3) $$\frac{(la+mc+ne)^3}{(lb+md+nf)^3}=\frac{(c+e)(e+a)(a+c)}{(d+f)(f+b)(b+d)};$$

Shew that these are illustrations of the theorem: "*If two or more fractions are equal, then any fraction whose numerator is a homogeneous expression in the numerators, and denominator an expression similarly formed of the denominators, is equal to any other fraction whose numerator is homogeneous and of the same degree in the numerators, and denominator similarly formed of the denominators.*"

9. If
$$\frac{l}{b-c}=\frac{m}{c-a}=\frac{n}{a-b},$$
then
$$l+m+n=0$$

10. If
$$\frac{x+y}{x-y}=\frac{y+z}{5(y-z)}=\frac{z+x}{7(z-x)}$$
then
$$20x+21y+6z=0$$

and of the quantities x, y, z, supposed real, two are of one sign and the remaining one of opposite sign.

11. If
$$\frac{x}{y+z}=\frac{y}{z+x}=\frac{z}{x+y}$$
then $x=y=z$, unless $x+y+z=0$.

12. If
$$\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$$
then
$$(b-c)\,x+(c-a)\,y+(a-b)\,z=0.$$

13. If
$$\frac{a}{x^2-yz}=\frac{b}{y^2-zx}=\frac{c}{z^2-xy}$$
then
$$ax+by+cz=(a+b+c)\,(x+y+z).$$

14. If
$$\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$$
then
$$(a+b+c)(yz+zx+xy)=(x+y+z)(ax+by+cz).$$

15. If
$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c} \text{ then}$$
$$\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}=\frac{(x+y+z)^3}{(a+b+c)^2}.$$

16. If
$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c},$$
then
$$\frac{x^2+a^2}{x+a}+\frac{y^2+b^2}{y+b}+\frac{z^2+c^2}{z+c}=\frac{(x+y+z)^2+(a+b+c)^2}{(x+y+z)+(a+b+c)}.$$

17. Solve
$$x-5y+2z=0,$$
$$7x+9y-6z=0,$$
$$yz-2zx+xy=4.$$

18. Solve
$$7x-11y+z=0,$$
$$13x+5y-49z=0,$$
$$3x^2-5y^2-z^2=x+2y-z.$$

19. Solve
$$yz-3zx+2xy=0,$$
$$5yz+6zx-18xy=0,$$
$$4x-5y+7z=21.$$

20. Shew that the three equations
$$x+y-z=0,$$
$$7x-9y+3z=0,$$
$$2x+5y-3z=0,$$
cannot all be true, except for the obvious zero solution $x=y=z=0$.

21. If $\frac{x}{a} = \frac{y}{b}$ then will $ax + by$ be a mean proportional between $x^2 + y^2$ and $a^2 + b^2$, and conversely.

22. If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ then will

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax + by + cz)^2$$

and conversely.

23. If $\frac{x}{a}$, $\frac{y}{b}$, $\frac{z}{c}$ are three ratios, not all equal, of positive quantities, and if l, m, n are positive, then the ratio

$$\frac{lx + my + nz}{la + mb + nc}$$

lies in value between the greatest and the least of the given ratios.

CHAPTER III

VARIATION

1. **Preliminary.** In the applications of algebra there frequently present themselves quantities which are undergoing or may be supposed to undergo change, so that the numbers which measure them are variables. Any problem concerned with such variable quantities will involve two or more variable numbers, and these numbers are related so that a change in any one will cause at least one other number to change. That part of the subject which has to do with such related variables and the laws of their dependence on one another is called **variation.**

The following illustrations should be examined in order that the formal theorems may be more readily understood.

Illustration 1. An observer on a railway train notes that at a certain instant he is passing a mile-post; at the end of $1\frac{1}{2}$ minutes he notes that he passes the next mile-post, at the end of 3 minutes the next, at the end of $4\frac{1}{2}$ minutes the next, at the end of 6 minutes the next, and at the end of $7\frac{1}{2}$ minutes the next. What inference is to be drawn as to the motion of the train?

Let s (miles) be the distance travelled by the train in t (minutes) from the time when the observation began. As time passes t changes and s varies *with* t, or, in other words, a change in t necessitates a change in s.

Now let us take note of the observed facts. When t changes from $1\frac{1}{2}$ to 3, s changes from 1 to 2, and, as $\frac{1\frac{1}{2}}{3}=\frac{1}{2}$, the change in s is proportionate to the change in t; next when t changes from 3 to $4\frac{1}{2}$, s changes from 2 to 3, and as $\frac{3}{4\frac{1}{2}}=\frac{2}{3}$ the change in s is proportionate to the change in t, and so for the other given values of s and t. It thus appears that the change in s is proportionate to the change in t, and we infer that the train is moving uniformly.

When two variable quantities are so related that any change in the one implies a proportionate change in the other, then each is said to **vary as** the other.

Here s varies as t (or t varies as s), and this is written

$$s \propto t.$$

Illustration 2. A small bullet is allowed to fall freely from a height and at the end of 1 sec., 2 sec., 3 sec. it is found to have fallen through 16·1 ft., 64·4 ft., 144·9 ft. What seems to be the relation between the time measured from the instant the bullet was let fall and the distance through which it has fallen in that time?

Let t and s measure the time and distance in question. It is at once plain that s while *varying with* t does not vary as t. As t changes from 1 to 2, s changes from 16·1 to 64·4; now, $\frac{16 \cdot 1}{64 \cdot 4} = \frac{1}{4} = \frac{1^2}{2^2}$, so that the change in s is proportionate to the change in the square of t. So when t changes from 2 to 3, s changes from 64·4 to 144·9, and, as $\frac{64 \cdot 4}{144 \cdot 9} = \frac{4}{9} = \frac{2^2}{3^2}$, the change in s is proportionate to the change in the square of t. Hence, from the observations given we are led to suppose that s varies as the square of t. Investigations shew that this is the law of bodies falling freely. We say then that s varies as t^2, and write

$$s \propto t^2.$$

Illustration 3. The volume of a certain gas under pressure 14 (pounds to the square inch) is 108 (cubic feet); under pressure 21 the volume is found to be 72, and under pressure 28 the volume is found to be 54. In what way do the volume and the pressure seem to be related?

Let p and v denote the measures of the pressure and the volume.

When p changes from 14 to 21, v changes from 108 to 72; now 14 : 21 as 2 : 3, and 108 : 72 as 3 : 2 or as $\frac{1}{2} : \frac{1}{3}$ so that the change in the volume is proportionate to the change in the reciprocal or the inverse of the pressure. The same is found for the change from pressure 21

to pressure 28 so that when the pressures are as $2:3:4$ the volumes are as $\frac{1}{2}:\frac{1}{3}:\frac{1}{4}$. Here v is said to **vary inversely** as p and we write

$$v \propto \frac{1}{p}.$$

2. **Theorem.** The following theorem is the one fundamental proposition for the case in which appear only two variables of which one may be called the dependent, the other the independent variable.

If $y \propto x$, and if x be allowed to vary, then the value of y corresponding to any value taken by x is equal to the product of that value of x and some constant number, i.e., a number which, as x and therefore y change, does not change.

Here x and y are the measures of quantities so that we may speak of the ratio $x:y$ while it might not be permissible to speak of the ratio of the quantities measured by them.

Let x take *any* values $x_1, x_2, x_3, x_4, \ldots\ldots$ and let the corresponding values of y be $y_1, y_2, y_3, y_4, \ldots\ldots$

Then since $y \propto x$

$$\frac{x_1}{x_2} = \frac{y_1}{y_2}$$

$$\therefore \frac{y_1}{x_1} = \frac{y_2}{x_2}$$

Similarly $$\frac{y_1}{x_1} = \frac{y_3}{x_3} = \frac{y_4}{x_4} = \ldots\ldots$$

$\therefore$ since $$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} = \frac{y_4}{x_4} = \ldots\ldots$$

it follows that, as x and therefore y change, the ratio of y to x does not change, *i.e.*,

$$\frac{y}{x} = k \text{ (a constant)}$$

$$\therefore\ y = kx$$

as it was required to prove.

Conversely: *If y and x are two related variables and if, as x and therefore y change, $y = kx$ where k is a constant, then $y \propto x$.*

For let x_1, x_2 be *any* two values of x, and y_1, y_2 the corresponding values of y. Then by the condition given

$$y_1 = kx_1,$$
$$y_2 = kx_2$$

$\therefore$ by division
$$\frac{y_1}{y_2} = \frac{x_1}{x_2}.$$

Thus, *any* change in the value of x requires a proportionate change in the value of y, *i.e.*, $y \propto x$.

Ex. 1. The velocity of a particle falling freely from rest varies as the time from rest. At the end of $1\frac{1}{2}$ seconds the velocity is observed to be 48·3 feet a second. Find the relation giving the velocity in terms of the time, and the velocity at the end of 2 seconds.

Let v (feet a second) be the velocity at the end of time t (seconds) from rest. Then by the given condition

$$v \propto t$$
$$\therefore \; v = kt$$

where k is a constant whose value is to be found.

If $t = 1\frac{1}{2}$, v is 48·3.

$$\therefore \; 48{\cdot}3 = k \times 1\tfrac{1}{2}$$
$$\therefore \; k = 32{\cdot}2.$$

The value of k having been found we may write

$$v = 32{\cdot}2t$$

as the relation giving v in terms of t.

If $t = 2$, $v = 32{\cdot}2 \times 2 = 64{\cdot}4$, and the velocity at the end of 2 seconds is 64·4 feet a second.

Ex. 2. The space through which a particle falls freely from rest is known to vary as the square of the time from rest. In 1 second a particle is observed to fall 16·1 feet. Find the relation connecting the space and the time, the space through which the particle would fall in 3 seconds, and the space through which it would fall in the third second.

Let t and s measure the time in seconds and the space in feet from rest. Then

$$s \propto t^2$$
$$\therefore \; s = kt^2$$

where k is a constant whose value is not yet known.

If $t=1$, $s=16{\cdot}1$.

$$\therefore\ 16{\cdot}1=k\times 1^2$$
$$\therefore\quad k=16{\cdot}1.$$

Thus the relation connecting s and t is

$$s=16{\cdot}1t^2.$$

If $t=3$, $s=16{\cdot}1\times 3^2=144{\cdot}9$, and the space through which the particle falls in 3 seconds is 144·9 feet.

Similarly, in 2 seconds the particle falls through $16{\cdot}1\times 2^2$ or 64·4 feet. Therefore in the third second the particle falls through $144{\cdot}9-64{\cdot}4$ or 80·5 feet.

Ex. 3. The volume of a gas under change of pressure varies inversely as the pressure. The volume of a certain quantity of gas under pressure 15 (pounds on the square inch) is 96 (cubic feet). Find the volume under a pressure of 18.

Let v and p measure the volume and the pressure in the units indicated. Then

$$v\propto\frac{1}{p}$$
$$\therefore\ v=k.\frac{1}{p}$$

where k is a constant whose value is not yet known.

If $p=15$, $v=96$.

$$\therefore\ 96=k.\frac{1}{15}$$
$$\therefore\quad k=15\times 96$$
$$\therefore\quad v=15\times 96.\frac{1}{p}.$$

Therefore, if $p=18$, $v=15\times 96\times\dfrac{1}{18}=80$ and the volume of the gas under pressure 18 is 80 cubic feet.

EXERCISES

1. The area of a circle is known from geometry to vary as the square of the radius. The area of a circle of radius $3\frac{1}{2}$ is found to measure 38·5. Find the formula for the area of a circle.

2. The surface of a sphere is known to vary as the square of the radius. A sphere of radius $1\frac{3}{4}$ is found to have a surface area of 38·5. Find the formula for the area of the surface of a sphere.

3. The volume of a sphere varies as the cube of its radius. Metal spheres of radii 3, 4, 5 are melted and cast into a single sphere. Find its radius.

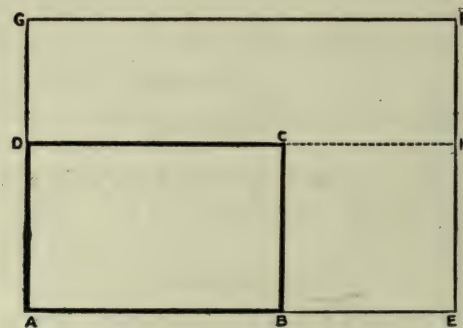
4. The *illumination* from a source of light varies inversely as the square of the distance. At what distance from the source must a small sheet of paper be placed to receive $3\frac{3}{8}$ times as much light as when at a distance 15 feet from the source? (Why a *small* sheet?)

5. The square of the time of a planet's revolution about the sun varies as the cube of its mean distance from the sun. The mean distances of Venus and the Earth being given as in the ratio of 18 : 25 find in days the time of revolution of Venus.

6. If x and y vary in such a way that their product is constant, then y varies inversely as x or x varies inversely as y.

3. **Problems Involving more than Two Variables.** Up to this point only two variables have appeared in any problem. In the following example appear three variables:

Ex. From geometry it is known that the area of a rectangle depends only on its base and its altitude; it is known also that the area varies as the base when the altitude is constant and as the altitude when the base is constant. It is required to study the variation in the area when both the base and the altitude vary.



Let A, b, h measure the area, the base and the height of a rectangle ABCD. Let b change to *any* value $b'(=\text{AE})$ and h to *any* value $h'(=\text{AG})$, and *as a result* suppose A to change to A′. We seek a relation between the old values A, b, h, and the new values A′, b', h'.

First, suppose b to change to b', h not changing, so that the rectangle ABCD becomes the rectangle AEHD, and let the area of AEHD be A_1. In this change the base has changed, the altitude not changing, and therefore by what is given

$$\frac{A}{A_1}=\frac{b}{b'} \qquad \text{(I)}.$$

Next, suppose h to change to h', b' not changing, so that the rectangle AEHD becomes the rectangle AEFG. In this change the altitude has changed, the base not changing, and therefore by what is given

$$\frac{A_1}{A'}=\frac{h}{h'} \qquad \text{(II)}.$$

Therefore from (I) and (II), by multiplication,

$$\frac{A}{A_1}\cdot\frac{A_1}{A'}=\frac{b}{b'}\cdot\frac{h}{h'}$$

or

$$\frac{A}{A'}=\frac{bh}{b'h'}.$$

Thus, if b and h both change in any way the change in the *product* bh is proportionate to the change in A and therefore A varies as bh, or the area varies as the product of (the measures of) the base and the altitude.

The preceding is a simple illustration of the following theorem, the proof of which may now be briefly stated.

Theorem. *If x is variable depending for its value on the two variables y and z, which are independent of each other, and if $x \propto y$ when z is constant, and $x \propto z$ when y is constant, then $x \propto yz$ when y and z both vary.*

First, let y take *any* value y_1, and z take any value z_1: denote the resulting value of x by x_1. Then let y and z take *any* new values y_2 and z_2: denote the resulting value of x by x_2. The relation between the old and the new values of x, y, z is sought.

In order to use the given conditions, suppose, when y changes from y_1 to y_2, that z retains the value z_1: denote the resulting value of x by x'.

Then (x_1, y_1, z_1), (x', y_2, z_1), (x_2, y_2, z_2) are three sets of corresponding values of the variables.

In passing from the first to the second set z is constant. Hence

$$\frac{x_1}{x'} = \frac{y_1}{y_2} \qquad \text{(I).}$$

In passing from the second to the third set y is constant. Hence

$$\frac{x'}{x_2} = \frac{z_1}{z_2} \qquad \text{(II).}$$

Therefore, from (I) and (II), by multiplication,

$$\frac{x_1}{x_2} = \frac{y_1\ z_1}{y_2\ z_2}$$

which means that when the variables pass from any one possible set of values to any other, the change in x is proportionate to the change in the *product* yz. Hence $x \propto yz$.

When x varies as the product of two or more variable quantities it is said to vary **jointly** as those quantities.

Further complexities in variation will be developed in the exercises and examples which follow.

Ex. 1. The area of a rectangle is known from geometry to vary as the base when the altitude is constant and as the altitude when the base is constant. The area of a square of side 1 being the unit of area find the formula for the area of a rectangle.

Let A, b, h measure the area, the base and the altitude of the rectangle. Then $A \propto b$ when h is constant and $A \propto h$ when b is constant and b and h are independent.

$$\therefore\ A \propto bh$$

$$\therefore\ A = k.bh,$$

where k is a constant, *i.e.*, the same for all rectangles.

If $b=1$ and $h=1$, then $A=1$

$$\therefore\ 1 = k.1 \times 1$$

$$\therefore\ k = 1$$

$$\therefore\ A = bh,$$

the formula required.

Ex. 2. It is shewn in works on geometry that the volume of a cone varies as the square of the radius of the base when the altitude is constant and as the altitude when the base is constant. A cone of height 1 and with a base of radius 1 is measured and found to be $\frac{22}{21}$. Find the formula for the volume of a cone.

Let v, h, r measure the volume, the altitude and the radius of the base of the cone. Then

$$v \propto r^2 \text{ when } h \text{ is constant}$$

also

$$v \propto h \text{ when } r^2 \text{ (or } r) \text{ is constant}$$

$$\therefore\ v \propto hr^2$$

$$\therefore\ v = k.hr^2,$$

where k is constant.

If $r=1$ and $h=1$, we are given that $v=\frac{22}{21}$

$$\therefore\ \tfrac{22}{21} = k.1 \times 1^2$$

$$\therefore\ k = \tfrac{22}{21}$$

$$\therefore\ v = \tfrac{22}{21} hr^2,$$

the formula required.

Ex. 3. The volume of a gas depends upon the temperature and the pressure. In works on physics it is given that the volume varies as the absolute temperature when the pressure is kept constant and inversely as the pressure when the temperature is kept constant. The volume of a certain quantity of gas at absolute temperature 273 and under pressure 14 (pounds on the square inch) is 78 (cubic feet); find its volume at temperature 300 and under pressure 20.

Let v, t, p measure the volume of the gas, the absolute temperature and the pressure. Then

$$v \propto \frac{1}{p}, \text{ when } t \text{ is constant};$$

also

$$v \propto t, \text{ when } \frac{1}{p} \text{ (or } p) \text{ is constant.}$$

$$\therefore\ v \propto \frac{1}{p} . t$$

$$\therefore\ v = k . \frac{t}{p}$$

where k is constant for the quantity of gas in question.

If $t=273$ and $p=14$ it is given that $v=78$.

$$\therefore\ 78=k\,.\,\frac{273}{14}.$$

$$\therefore\quad k=4.$$

$$\therefore\quad v=4\,.\,\frac{t}{p}.$$

Therefore if $t=300$ and $p=20$,

$$v=4\,.\,\frac{300}{20}=60.$$

The volume required is then 60 cubic feet.

In all problems like the preceding, in which the value of the constant is found, it is to be noted that, when magnitudes of different kinds appear, the value found depends upon the choice of the units of measurement.

EXERCISES

1. The volume of a pyramid is shewn in works on geometry to vary as (the area of) the base when the altitude is constant, and as the altitude when the base is constant. A certain pyramid whose base and altitude measure 18 and 5 is found to have 30 as the measure of its volume. Find the formula for the volume of a pyramid in terms of its base and altitude.

2. If v, h, r measure the volume, the altitude and the radius of the base of a cylinder, it is known that $v \propto h$ if r is constant, and $v \propto r^2$ if h is constant. For a cylinder in which $r=2$ and $h=5$ it is found that $v=62{\cdot}8320$. Find the formula for the volume of a cylinder, and also the volume of a cylinder for which $r=3$, $h=7$.

3. The weight of a coin of gold alloy varies as the square of its diameter, and as its thickness. When the thickness is 0·1cm. and the diameter is 2cm., the weight is 5·91g; find the weight of a coin of the same alloy of thickness 0·2cm. and diameter 3cm.

4. Given that y is equal to the sum of two quantities of which one is constant and the other varies as x, and that $y=1$ when $x=1$, while $y=-1$ when $x=2$, find

(1) y when $x=0{\cdot}5$;
(2) x when $y=0$.

5. If $x \propto \frac{1}{y}$ when z is constant, and $x \propto \frac{1}{z}$ when y is constant, where y and z are independent, then when y and z both vary $x \propto \frac{1}{yz}$.

EXAMPLES

1. The velocity (v) of a heavy particle falling freely from rest varies as the square root of the distance (s) through which the particle has fallen. If when $s = 16{\cdot}1$ (ft.) it is found that $v = 32{\cdot}2$ (ft. a sec.) find the equation connecting v and s.

2. The electrical resistance of a wire varies directly as the length of the wire and inversely as the square of the radius of the cross section. If, when a copper wire is 1m. long and 1mm. in diameter, the resistance is 0·02 ohms, find the resistance of a copper wire 7m. long and 1·5mm. in diameter.

3. It is shewn in works on geometry that the volume of a pyramid varies as its base when its height is constant and as its height when its base is constant. A cube may be divided into six equal pyramids, each having a face of the cube as base, and the centre of the cube as vertex.

From these facts obtain the general formula for the volume of a pyramid.

4. The value of w depends only on the values of the three independent variables x, y, z. If it is known that $w \propto x$ when y and z are constant, that $w \propto y$ when z and x are constant, and that $w \propto z$ when x and y are constant, shew that $w \propto xyz$, when x, y and z vary.

Illustrate by reference to the volume of a rectangular parallelopiped.

5. If $y \propto x$ shew that $x^2 + y^2 \propto xy$ and construct an illustration.

6. If x varies as y and inversely as the square root of z, and if, when y is 5 and z is 4, the value of x is 15, find the value of x when $y = 7$ and $z = 9$.

7. If y varies as the sum of two quantities, of which one is constant and the other varies as the square of x, and if $y = 4$ when $x = 1$, and $y = 28$ when $x = 5$, find y when $x = 3$.

8. Given that y varies as the sum of two variables, one of which varies as x and the other inversely as x, and that $y = 19$ when $x = 3$, and that $y = 11$ when $x = 2$, find y when $x = 1$.

9. If y equals the sum of three quantities of which one is constant, one varies as x, and the other inversely as x, and if $(y = 9,\ x = 1)$, $(y = 8,\ x = 2)$, $(y = 9,\ x = 3)$ are three sets of corresponding values, find y when $x = 4$.

10. When a weight is hung by an elastic string, the *extension* (*i.e.*, the amount the string is stretched) varies as the weight and as the unstretched length of the string. If a weight of 3·5kg. stretches a string 1m. long to a length 1·1m., to what length will a weight of 4·5kg. stretch a string of the same kind of length 1·5m.?

11. When a weight is hung by an elastic string of given material, the extension varies as the weight, as the unstretched length of the string, and inversely as the square of the diameter of the string. If a weight of 8 lb. stretches a string 2·5 ft. long and of $\frac{1}{4}$ in. diameter to a length 2·7 ft., to what length will a weight of 10 lb. stretch a string of the same material 3·25 ft. long and of $\frac{1}{6}$ in. diameter?

12. If $x^2 + y^2 \propto xy$ shew that $x \propto y$, and if, when $x^2 + y^2 = 25$ it is known that $xy = 12$, find the relation between x and y.

13. If y varies inversely as x, and if when $y = 2$ it is known that $x = 1$, construct a graph to exhibit related values of x and y.

CHAPTER IV

SCALES OF NOTATION

1. **Explanatory.** In the ordinary method of writing numbers a symbol as 3674 means

$$3 \times 1000 + 6 \times 100 + 7 \times 10 + 4,$$

or,

$$3.10^3 + 6.10^2 + 7.10 + 4$$

and the number is said to be written in the **scale of ten.** Scales other than of ten have been in use, notably the scale of twelve in which the numbers 2, 3, 4, 6,—factors of twelve—enjoy an advantage, in computations, similar to that belonging to the numbers 2 and 5 in the ordinary scale.

It is to be noticed that in the scale of ten there are employed nine digits and the cipher or zero. In the scale of twelve for example there will be needed eleven digits and the zero; for these, the nine digits of the ordinary scale will be employed in their accepted meaning, and two additional symbols, say t for ten and e for eleven, so that

$$4e5t$$

in the scale of twelve means

$$4.12^3 + 11.12^2 + 5.12 + 10$$

these latter numbers being read in the ordinary scale.

Thus it is seen that to *one number* there correspond *many modes of expression* as different scales are taken. For example, 45 in the scale of ten and 39 in the scale of twelve denote the same number.

A few problems and theorems will be treated as having an interest in themselves or as bringing into mind the significance of a notation.

2. **Transformation** or **Change of Scale.** The first problem that presents itself is that of expressing in a new scale a number given in a stated scale. For the present the stated scale will be the scale of ten.

The number 27, in the scale of ten, if we propose to express it in the scale of twelve, is seen to be $2 \times 12 + 3$, and would be written 23 in that scale. In like manner 557, in the scale of ten is seen to be $46 \times 12 + 5$: here, however, the multiplier of 12 is greater than 12, and in any scale, there is no single symbol for a number equal to or greater than the number giving the scale. Accordingly since $46 = 3 \times 12 + 10$, it follows that

$$\begin{aligned} 557 &= (3 \times 12 + 10) \times 12 + 5, \\ &= 3.12^2 + 10.12 + 5, \\ &= 3t5, \text{ in the scale of twelve.} \end{aligned}$$

The method is sufficiently indicated in these illustrations and the working process is shewn in the following example, it being proposed to express, the scale of twelve, the number 8593 given in the scale of ten:

$$\begin{array}{r@{\;}l} 12\underline{)8593} & \\ 12\underline{)716} & : \ 1 \\ 12\underline{)59} & : \ 8 \\ 4 & : \ 11 \end{array}$$

which warrants the statement

$$\begin{aligned} 8593 &= 4.12^3 + 11.12^2 + 8.12 + 1 \\ &= 4e81 \text{ in the scale of twelve.} \end{aligned}$$

Further, as *decimal fractions* are an essential part of the notation in the scale of ten, *radix-fractions* present themselves naturally in any named scale. Thus 0·73, given as in the scale of twelve, means $\frac{7}{12} + \frac{3}{12^2}$. There arises then the problem of expressing a given decimal fraction as a radix-fraction in a given scale. Fractions less than unity need alone be considered. Consider first two simple illustrations.

(1) $0{\cdot}75 = \frac{3}{4} = \frac{9}{12}$ so that 0·75 in the scale of ten equals 0·9 in the scale of twelve. Note that this result might have been obtained by solving the equation

$$0{\cdot}75 = \frac{x}{12}$$

since the solution turns out to be an integer.

(2) It is easy to verify that

$$0{\cdot}5625 = \frac{6}{12} + \frac{9}{12^2}$$

$$= 0{\cdot}69 \text{ in the scale of twelve,}$$

and this is involved in the equation,

$$0{\cdot}5625 = \frac{x}{12} + \frac{y}{12^2}$$

since this admits a solution $x = 6$, $y = 9$, numbers which are integers and less than the number giving the scale. The question of finding the required solution of this equation arises. If both sides be muliplied by 12, we have

$$6 + 0{\cdot}75 = x + \frac{y}{12}\cdot$$

Then since x must be an integer, and $\frac{y}{12}$ a proper fraction, it must be that $x = 6$, that $\frac{y}{12} = 0{\cdot}75$, or $y = 9$. Hence

$$0{\cdot}5625 = \frac{6}{12} + \frac{9}{12^2} = 0{\cdot}69 \text{ in the scale of twelve.}$$

The illustrations suggest a method. Let it be required to express $0{\cdot}275$, given in the scale of ten, as a radix-fraction in the scale of 12. Put

$$0{\cdot}275 = \frac{x}{12} + \frac{y}{12^2} + \frac{z}{12^3},$$

where on the right it seems to be assumed that only three number symbols will be needed. Note that x, y, z are to be less than 12, and we are to determine them as integers. Multiplying through by 12 we have

$$3 + 0{\cdot}3 = x + \frac{y}{12} + \frac{z}{12^2}.$$

Plainly $\frac{y}{12} + \frac{z}{12^2}$ is less than unity, so that if x is an integer it must be **3**. Then it follows that

$$0{\cdot}3 = \frac{y}{12} + \frac{z}{12^2}$$

and multiplication by 12, yields the result

$$3{\cdot}6 = y + \frac{z}{12}$$

so that $y = 3$ and $\frac{z}{12} = 0{\cdot}6$. But this gives z as equal to $7{\cdot}2$, and we have the fact

$$0{\cdot}275 = \frac{3}{12} + \frac{3}{12^2} + \frac{7{\cdot}2}{12^3}.$$

This is not strictly a radix fraction, because $7{\cdot}5$ is not an integer, yet the essential idea here finds expression. If more than three number symbols x, y, z, had been taken we should have had more terms of the kind demanded.

The working process is suggested. It is a process of *continued multiplication, the integral part of each product being dropped*, just as in the earlier problem of changing the scale for an integral number the process was one of *continued division, the remainder in each division being dropped.*

This process may be exhibited thus:

$$\begin{array}{r}
0{\cdot}275 \\
12 \\
\hline
3 + 0{\cdot}3 \\
12 \\
\hline
3 + 0{\cdot}6 \\
12 \\
\hline
7 + 0{\cdot}2 \\
12 \\
\hline
2 + 0{\cdot}4 \\
\textit{etc.}
\end{array}$$

and we write $0{\cdot}275 = 0{\cdot}3372 \ldots\ldots$ in the scale of twelve. It can easily be seen that the process will not terminate, but the conversion of vulgar fractions to decimals involves the same difficulty.

EXERCISES

1. Transform from the scale of ten
   (i) 3759 to the scale of eight;
   (ii) 5387 to the scale of eleven;
   (iii) 6937 to the scale of five;
   (iv) 1237 to the scale of two.

2. Express as radix-fractions
   (i) 0·25, radix eight;
   (ii) 0·37, radix seven;
   (iii) 0·315, radix five;
   (iv) 0·23, radix eleven.

3. Transform from the scale of ten
   (i) 137·5 to the scale of twelve;
   (ii) 397·8 to the scale of five;
   (iii) 519·23 to the scale of nine;
   (iv) 317·25 to the scale of seven.

4. The number 96 in a certain scale is the same as 87 in the ordinary notation. Find the scale.

5. The number 456 in a certain scale is the equivalent of 237 in the ordinary notation. Find the scale.

2. Operations.

(1) *Addition*: Find the sum of 3759 and $4e58$ in the scale of twelve, e denoting the number eleven.

$$\begin{array}{r} 3759 \\ 4e58 \\ \hline 86e5 \end{array}$$

Here when 9 and 8 are added to give what is written 17 in the common scale, we regard the 17 as $12 + 5$, and would write it 15, the scale being 12, so that there is 5 *and* 1 *to carry*, etc.

(2) *Subtraction*: Subtract 3425 from 5623, the numbers being in the scale of seven.

$$\begin{array}{r} 5623 \\ 3425 \\ \hline 2165 \end{array}$$

(3) *Multiplication*: Find the product of 675 and $5\,t\,4$, the scale being eleven and t denoting ten.

$$\begin{array}{r} 675 \\ 5t4 \\ \hline 2479 \\ 6086 \\ 3043 \\ \hline 367529 \end{array}$$

(4) *Division*: Divide 76426 by 547, the numbers being written in the scale of eight.

```
547)76426(131
    547
    ----
    2152
    2065
    ----
     656
     547
     ---
     107
```

Here, at the second partial division, the part of the quotient is found, not by saying "five into twenty-one, for 21 is not twenty-one," but by saying "five into twice eight plus one or seventeen."

EXERCISES

1. Perform the following operations:
   (i) $3875+4586+3157$ in the scale of nine;
   (ii) $6te4-3et8$ in the scale of twelve;
   (iii) 2364×2165 in the scale of seven;
   (iv) $39t4e\div 13e$ in the scale of twelve.

2. Find the value in the stated scale of
   (i) $3172+138e-2t48$ in the scale of eleven;
   (ii) $(2563+1274)\times(5137-3676)$ in the scale of eight;
   (iii) $t378t03\div(386t-2e79)$ in the scale of twelve;
   (iv) $325\times 731\times 154$ in the scale of nine.

3. Find the value of
   (i) $23{\cdot}57+3{\cdot}26+5{\cdot}48$, the scale being nine;
   (ii) $3{\cdot}56+2{\cdot}7t-3{\cdot}9e$, the scale being twelve;
   (iii) $12{\cdot}37\times 4{\cdot}65$, the scale being eight;
   (iv) $76{\cdot}325\div 4{\cdot}57$, the scale being nine.

4. Express
   (i) in the scale of six, 5432 given in the scale of eight;
   (ii) in the scale of twelve, 3724 given in the scale of nine;
   (iii) in the scale of ten, *te*74 given in the scale of twelve;
   (iv) in the scale of ten, 6351 given in the scale of seven.

5. Express in the scale of seven the numbers 347 and 569 given in the scale of ten, find their product in the new scale and express the result in the scale of ten.

Verify by multiplying the numbers as given in the scale of ten.

6. Find the square root, working in the scale given, of

(i) 562 in the scale of seven;
(ii) 108845 in the scale of eleven;
(iii) 11234 in the scale of five;
(iv) 15240 in the scale of nine.

Verify in each case by squaring the result, and also by transforming the given numbers to the scale of ten, extracting the root and changing it to the original scale.

7. Find in what scale the product of 123 and 3 is 424.

8. Shew by continued division by $x-2$ that

$$3x^3-13x^2+9x+23=3(x-2)^3+5(x-2)^2-7(x-2)+13.$$

9. Express in powers of $x-3$ the polynomial $8x^4-7x^3+11x^2+12x-19$.

4. Theorems.

I. *An integer expressed in scale r when divided by $r-1$ (or any factor of $r-1$) will leave the same remainder as the sum of the digits divided by $r-1$ (or by that factor of $r-1$).*

Let N be the number and $p_0, p_1, \ldots p_n$ the digits in reversed order so that

$$\mathrm{N}=p_0+p_1r+p_2r^2+\ldots+p_nr^n.$$

$$\therefore\ \mathrm{N}=(p_0+p_1+\ldots+p_n)+p_1(r-1)+p_2(r^2-1)+\ldots+p_n(r^n-1).$$

$\therefore$ Then dividing each side by $r-1$, we have since $(r^n-1)\div(r-1)=(r^{n-1}+r^{n-2}+\ldots+1)$

$$\frac{\mathrm{N}}{r-1}=\frac{p_0+p_1+\ldots+p_n}{r-1}+\left[p_1+p_n(r+1)+\ldots\ \ldots\right]$$

The part within the brackets [ ] is an integer. Therefore

$$\frac{\mathrm{N}}{r-1}\text{ and }\frac{p_0+p_1+\ldots+p_n}{r-1}\quad\text{must}$$

yield the same fractional part. Therefore the division of N and of $p_0+p_1+\ldots+p_n$ by $r-1$ must yield the same remainder.

Next let $r-1=st$. Then multiplying the last equality by t we have

$$\frac{\mathrm{N}}{s}=\frac{p_0+p_1+\ldots+p_n}{s}+\text{ an integer},$$

so that N and $p_0+p_1+\ldots+p_n$ when divided by s must yield the same remainder.

II. *An integer expressed in the scale r is divisible by $r+1$ if the difference between the sum of the digits in the odd places and the sum of the digits in the even places is divisible by $r+1$.*

Let N be the number and $p_0, p_1, \ldots p_n$ the digits in the order right to left. Then

$$\begin{aligned} N &= p_0 + p_1 r + p_2 r^2 + \ldots + p_n r^n \\ &= (p_0 - p_1 + p_2 - p_3 + \ldots) \\ &\quad + p_1(r+1) - p_2(r^2-1) + p_3(r^3+1) - \ldots \end{aligned}$$

Now $r+1$, r^3+1, r^5+1, ... are divisible by $r+1$ giving quotients 1, r^2-r+1, $r^4-r^3+r^2-r+1, \ldots$ and r^2-1, $r^4-1, \ldots$ are divisible by $r+1$ giving quotients $r+1$, $r^3+r^2+r+1, \ldots$ and the quotients all are therefore integers.

$$\therefore \frac{N}{r+1} = \frac{p_0 - p_1 + p_2 - p_3 \cdots}{r+1} + \text{an integer.}$$

$$= \frac{(p_0 + p_2 + p_4 + \ldots) - (p_1 + p_3 + \ldots)}{r+1} + \text{an integer.}$$

Thus, if N is divisible by $r+1$, so also is the difference between the sum of the digits in the even places and the sum of the digits in the odd places.

In the statement of these theorems "digit" is taken, on account of brevity, to mean the number designated by the digit.

EXERCISES

1. State the two theorems in the particular case when $r = 10$.

2. Without actual division shew that 837259 when divided by 9 leaves the remainder 7.

3. Shew that 7358 is of the form $9m+5$ and that 6547 is of the form $9n+4$ and therefore that 7358×6547 is of the form $9p+2$ where m, n, p are integers.

Hence shew that 48072826 cannot be the product, whereas 48172826 may be the product.

4. Find the following products and check results by "*casting out nines*":

$$2173 \times 547 \text{ ; } 3549 \times 1234 \text{ ; } 2917 \times 3456.$$

5. Find the product of 7938 and 9185, and noting that one of the numbers is divisible by 11, check the result.

CHAPTER V

SERIES

An examination of each of the following sequences of numbers

$$1,\ 2,\ 3,\ 4,\ 5, \ldots .$$

$$3,\ 7,\ 11,\ 15,\ 19, \ldots .$$

$$3,\ 6,\ 12,\ 24,\ 48, \ldots .$$

$$\frac{1}{1^2},\ \frac{1}{2^2},\ \frac{1}{3^2},\ \frac{1}{4^2},\ \frac{1}{5^2}, \ldots .$$

shews that the numbers in succession stand in a certain relation to the earlier numbers, in other words, are continued in accordance with a *law* revealed by an examination of the earlier numbers. If the numbers of such a sequence are connected by the signs + or – 1 as in

$$3 + 7 + 11 + 15 + 19 + \ldots .$$

they constitute a **series** and the individual numbers are called the **terms** of the series.

I

ARITHMETICAL SERIES

1. **Definition.** It is readily seen that

$$2 + 5 + 8 + 11 + \ldots ..$$

is a series, each term being formed from the preceding term by the addition of 3. Thus, consecutive terms differ by the same number, or, in other words, the difference between consecutive terms is constant. Such a series is called an **arithmetical series,** or an **arithmetical progression** which may, therefore, be defined as follows:

An arithmetical progression is a series in which each term is formed from the preceding by the addition of the same quantity.

All arithmetical series are seen to be included in the *general* progression

$$a+(a+d)+(a+2d)+(a+3d)+\ldots\ldots$$

Here a is called the **first term** and d (the difference between consecutive terms) the **common difference.**

Any term of a given arithmetical progression may be found without constructing all preceding terms. Thus, in the series

$$2+5+8+11+\ldots\ldots$$

in which 2 is the first term and 3 the common difference, it is readily seen that the *fiftieth* term, being *forty-nine* terms in advance of the first, is $2+3\times 49$ or 149, since each advance implies the addition of 3. So, too, the nth term of this series (n, a positive integer), being $n-1$ terms in advance of the first term, is $2+(n-1)3$ or $3n-1$. The nth term is usually spoken of as the **general term,** since from it any term can be found by giving to n a suitable value. Thus, here, the nth term being $3n-1$, the 5th term, for example, is $3\times 5-1$ or 14. In like manner we have for the general series,

$$nth\ term=a+(n-1)d \qquad (I)$$

a result which should be remembered.

Ex. 1. The 5th term of an arithmetical progression is 19 and the 13th term is 43; find the progression and the 30th term.

Let a, d denote the first term and the common difference

$$\therefore \text{ The 5th term}=a+4d$$

$$\therefore \qquad a+4d=19.$$

Similarly, by constructing the 13th term we have

$$a+12d=43.$$

$$\therefore \text{ solving, } a=7,\ d=3.$$

Thus the series is

$$7+10+13+16+19+\ldots\ldots$$

and the 30th term $=7+3\times 29$ or 94.

Ex. 2. The nth term of a series is $3n+5$; shew that the series is an arithmetical progression.

Here n is any number and we may say that the 1st term is $3\times 1+5$ or 8, the 2nd term is $3\times 2+5$ or 11, the 3rd term is $3\times 3+5$ or 14, the 4th term is $3\times 4+5$ or 17, etc. It *appears* then that the series is an arithmetical

progression of which the first term is 11 and the common difference is 3. But however many terms we construct we cannot say absolutely, on the evidence afforded by these terms, that for terms not constructed the law continues.

We may, however, reason generally and more briefly thus :

The nth term $=3n+5$, whatever value be assigned to n.

$\therefore$ the $(n-1)$th term $=3(n-1)+5$.

Then the difference between the nth and the $(n-1)$th term is equal to

$$(3n+5)-3(n-1)-5 \text{ or } 3.$$

Now n is *any* number ; therefore the difference between *any* and therefore *every* two consecutive terms of the series is 3, so that the series is an arithmetical progression.

EXERCISES

(NOTE : In the exercises, A.P. will be employed as an abbreviation for arithmetical progression.)

1. Find the 47th and the nth term of the following series :

(1) $1+3+5+7+\ldots\ldots$

(2) $7+13+19+25+\ldots\ldots$

(3) $29+25+21+17+\ldots\ldots$

(4) $94+81+68+55+\ldots\ldots$

In each case after finding the nth term test the result by finding from it the first four terms.

2. The 7th term of an A.P. is 23, and the 15th term -17 ; find the series, its 23rd term, and its first negative term.

3. Find three numbers in A.P. such that the third is 7 times the first while the product of the first and the third exceeds 5 times the second by a quantity equal to the first.

4. How many multiples of 13 are there between 300 and 700 ?

5. Shew that the series formed by taking every fifth term of the A.P.

$$1+3+5+7+\ldots\ldots$$

is also an A.P.

6. Shew that the series formed by taking every seventh term of the A.P.

$$a+(a+d)+(a+2d)+(a+3d)+\ldots\ldots$$

is also an A.P.

7. If to each term of an A.P. there be added the same number, the resulting series is an A.P.

8. If each term of an A.P. be multiplied by the same number the resulting series is an A.P.

9. Write down any 10 numbers in A.P. and shew that the average of the 1st and the 10th is the same as that of the 2nd and the 9th, or that of the 3rd and the 8th, etc.

10. Write down any 11 numbers in A.P. and shew that the average of the 1st and the 11th is equal to that of the 2nd and the 10th, or to that of the 3rd and the 9th, etc., and that this average is equal to the middle term.

11. The first of 17 terms of an A.P. is 5 and the last is 133; find the middle term.

12. The nth term of a series is $an+b$ whatever be the value of n; shew that the series is an A.P.

13. The pth term of an A.P. is q, and the qth term is p, where p and q are given integers; find the $(p+q)$th term.

2. Arithmetical Means. When three numbers are in arithmetical progression, the second number is called the **arithmetical mean** of the other two numbers. We may then have the problem:

To find the arithmetical mean of two given numbers.

Let a and b be the given numbers, and let x be the mean sought. Then by the definition

$$a, x, b$$

are in arithmetical progression, and therefore

$$x - a = b - x$$

$$\therefore \quad 2x = a + b$$

$$\therefore \quad x = \frac{a+b}{2}$$

so that *the arithmetical mean of two given numbers is equal to one-half their sum.*

In like manner we may have the problem of inserting any number of arithmetical means between two given numbers.

Ex. Insert 7 arithmetical means between 5 and 29.

Here 5, the 7 means sought, and 29 make up 9 terms in arithmetical progression. Let x be the common difference in this series of 9 terms. Then, the first term being 5, the ninth term equals $5+8x$, which must equal 29, *i.e.*,

$$5+8x=29$$
$$\therefore \quad 8x=24$$
$$\therefore \quad x=3$$

Thus the means are

$$8,\ 11,\ 14,\ 17,\ 20,\ 23,\ 26.$$

The general problem is:

To insert n arithmetical means between a and b.

Here a, the n means sought, and b make up $n+2$ terms in arithmetical progression. Let x be the common difference in this series of $n+2$ terms. Then, the first term being a, the $(n+2)$nd term is $a+(n+1)x$ which must be equal to b, *i.e.*,

$$a+(n+1)x=b$$
$$\therefore \quad (n+1)x=b-a$$
$$\therefore \quad x=\frac{b-a}{n+1}.$$

Thus the means are

$$a+\frac{b-a}{n+1},\ a+2\frac{b-a}{n+1},\ldots\ldots,a+(n-1)\frac{b-a}{n+1},\ a+n\frac{b-a}{n+1};$$

or, in simpler form,

$$\frac{na+b}{n+1},\ \frac{(n-1)a+2b}{n+1},\ \ldots\ldots\ \frac{2a+(n-1)b}{n+1},\ \frac{a+nb}{n+1}.$$

Since n has not any assigned value, it is not possible to write down all the means, but, the law of formation of successive means being known, we can regard all the means as found.

EXERCISES

1. Insert 11 arithmetical means between 19 and 223.

2. Insert 5 arithmetical means between 17 and -7.

3. When n arithmetical means are inserted between a and b find the rth mean.

4. Insert 5 arithmetical means between 11 and 25 and shew that the middle mean is the arithmetical mean of the given numbers.

5. Whatever odd number of arithmetical means be inserted between a and b shew that the middle mean is the arithmetical mean of a and b.

3. **The Summation Formula.** The sum of any number of terms of an arithmetical series may be found without actual addition.

Ex. Find the sum of 9 terms of the arithmetical progression,

$$13+17+21+\ldots\ldots$$

Let s denote the sum sought. Then

$$s=13+17+21+25+29+33+37+41+45$$
$$\therefore\ s=45+41+37+33+29+25+21+17+13,$$

the terms being written in reverse order, which does not affect the sum. Therefore by addition

$$2s=58+58+58+58+58+58+58+58+58$$
$$\therefore\ 2s=58\times 9$$
$$\therefore\ s=\frac{58\times 9}{2}=261.$$

It is to be noted that, in adding to find $2s$, when we find for the first addition $13+45$ the result 58, we know beforehand that the successive additions will yield the same sum, because in the upper line the terms *increase* 4 at each advance while in the lower line the terms *decrease* 4 at each advance. We know also beforehand that there will be 9 addends 58, *i.e.*, one for each term of the series. It follows then that it is not necessary to write all the terms of the proposed addition.

The general problem is the following:

To find the sum of n terms of the arithmetical progression

$$a+(a+d)+(a+2d)+\ldots\ldots$$

Let l denote the last of the terms considered, *i.e.*, the nth term

$$\therefore\ l=a+(n-1)d.$$

Let s denote the sum sought:

$$\therefore \quad s = a + (a+d) + (a+2d) + \ldots\ldots + (l-2d) + (l-d) + l.$$
$$\therefore \quad s = l + (l-d) + (l-2d) + \ldots\ldots + (a+2d) + (a+d) + a,$$

the terms being written in reverse order which does not affect the sum. Therefore, by addition,

$$2s = (a+l) + (a+l) + (a+l) + \ldots\ldots + (a+l) + (a+l) + (a+l),$$
$$\therefore \quad 2s = (a+l) \times n,$$

since there is one term $a+l$ for each term of the series.

$$\therefore \quad s = \frac{n}{2}(a+l) \qquad (II)$$

Further, putting for l its value in terms of a, d, n, we have

$$s = \frac{n}{2}\left\{2a + (n-1)d\right\} \qquad (III)$$

The formulæ (II) and (III) should be given verbal statement, and committed to memory.

The following examples may be examined:

Ex. 1. How many terms of the series

$$21 + 35 + 49 + \ldots\ldots$$

must be taken to yield the sum 336?

Let n be the number required. Then, the first term being 21 and the common difference being 14, we have, quoting the formula for the sum of n terms of an arithmetical series,

$$\frac{n}{2} \cdot \left\{2 \times 21 + (n-1)14\right\} = 336$$
$$\therefore \quad n(7n+14) = 336$$
$$\therefore \quad n^2 + 2n - 48 = 0$$

whence $n = 6$ or -8. Plainly the result must be a positive integer so that the number sought is 6.

It is to be remarked that if the 6 terms which yield the sum 336 be written down, and if, starting at the last term, we write down 8 terms in the inverse order of the series, the sum of these 8 terms is 336. This is not an *interpretation* of the negative root -8, but an indication of a closely related problem in which the number $+8$ is significant.

Ex. 2. Shew that the sum of $5n$ terms of an arithmetical progression is 5 times that of the n terms beginning with the $(2n+1)$th term.

Let a, d be the first term and the common difference of the series.

$$\therefore \text{ sum of } 5n \text{ terms} = \frac{5n}{2}.\left\{2a+(5n-1)d\right\} \qquad (1)$$

Also the $(2n+1)$th term $= a+2nd$

$\therefore$ sum of n terms beginning with the $(2n+1)$th

$$= \frac{n}{2}\cdot\left\{2(a+2nd)+(n-1)d\right\}$$

$$= \frac{n}{2}\cdot\left\{2a+(5n-1)d\right\} \qquad (2)$$

Then, comparing (1) and (2), we see that the result follows immediately.

EXERCISES

1. Sum to 53 terms, after the manner of the general theorem (*i.e.*, not quoting the formula), each of the following series:

(1) $5 + 11 + 17 + 23 + \ldots\ldots$

(2) $9 + 9\frac{3}{4} + 10\frac{1}{2} + 11\frac{1}{4} + \ldots\ldots$

(3) $117 + 112 + 107 + 102 + \ldots\ldots$

(4) $7\frac{1}{2} + 7\frac{1}{4} + 7 + 6\frac{3}{4} + \ldots\ldots$

(5) $\frac{a}{b} + \frac{a+b}{b} + \frac{a+2b}{b} + \ldots\ldots$

2. Sum to n terms, after the manner of the general theorem, each of the following series:

(1) $9 + 13 + 17 + 21 + \ldots\ldots$

(2) $23 + 18 + 13 + 8 + \ldots\ldots$

(3) $2\frac{1}{2} + 3\frac{3}{4} + 5 + 6\frac{1}{4} + \ldots\ldots$

In each case *test* the result by assigning to n the value 4.

3. Shew that

(1) $1+2+3+4+5 \ldots\ldots$ to n terms $= \frac{n(n+1)}{2}$;

(2) $1+3+5+7+ \ldots\ldots$ to n terms $= n^2$.

The first of these series is that of the first n natural numbers, the second that of the first n odd numbers. The results are important and should be carried in memory.

4. In the general A.P. of n terms,

$$a+(a+d)+(a+2d)+ \ldots\ldots +(a+\overline{n-1}d)$$

Shew that the sum of the rth term from the beginning and the rth term from the end is equal to the sum of the first and the last term.

Hence shew that one-half the sum of the first and the last term is the average of the terms and derive the formula for the sum of n terms of an A.P.

5. How many terms of the series

$$44+36+28+ \ldots\ldots$$

must be taken to yield the sum 128?

Comment on the two results.

6. How many terms of the series

$$21+17+13+9+ \ldots\ldots$$

must be taken to yield the sum 65?

Comment on the fractional result.

7. How many terms of the series

$$20+17+14+ \ldots\ldots$$

must be taken to yield the sum 76?

Comment on the fractional root.

8. How many terms of the series

$$18+22+26+\ldots\ldots$$

must be taken to yield the sum 210?

Comment on the negative root.

9. How many terms of the series

$$8+25+42+ \ldots\ldots$$

must be taken to yield the sum 1430?

Comment on the negative fractional root.

10. The nth term of a series is $5n+7$; shew that the series is an A.P., and find the sum of r terms.

11. The sum of 12 terms of an A.P. is 138 and the sum of 19 terms is 988; find the series and the sum of 26 terms.

12. Shew that the sum of an odd number of terms in A.P. is equal to the product of the middle term by the number of terms.

13. Find the sum of all multiples of 23 between 100 and 700.

14. The sum of n terms of a certain series is $7n^2+11n$, whatever be the value of n; find the rth term and shew that the series is an A.P.

15. Three integers are in A.P.; their sum is 24 and the product of the first and the last is 9 less than the square of the middle number. Find the numbers.

16. In a certain A.P. the first term is 29 and the last term 107; the sum of the terms is 952. Find the series and the number of terms.

17. Construct an A.P. such that the sum of 7 terms is equal to the sum of 11 terms, the common difference being 2.

18. If the pth, qth, rth terms of an A.P. are a, b, c respectively, shew that

$$(q-r)a+(r-p)b+(p-q)c=0.$$

19. A heavy particle, allowed to fall freely from a height, falls through 16·1 feet during the first second and in successive seconds falls through 32·2 feet more than during the preceding second. How far will it fall in t seconds?

II

GEOMETRICAL SERIES

1. **Definition.** It is readily seen that

$$2+6+18+54+\ldots\ldots$$

is a series, each term being formed from the preceding by multiplying it by 3. Thus consecutive terms stand to each other in a *constant* ratio. Such a series is called a **geometrical series,** or a **geometrical progression,** which may therefore be defined as follows:

A geometrical progression is a series in which each term is made from the preceding by multiplication by the same number.

All geometrical series are seen to be included in the general progression

$$a+ar+ar^2+ar^3+\ldots\ldots$$

Here a is called the **first term** and r (the ratio of any term to the preceding term) the **common ratio.**

Any term of a geometrical progression may be found without forming all preceding terms. Thus in the series

$$2+6+18+54+\ldots\ldots$$

in which 2 is the first term and 3 the common ratio, it is plain that the *fiftieth* term being *forty-nine* terms in advance of the first will be the product of 2 and the factor 3 taken forty-nine times, *i.e.*, will be 2.3^{49}, since each advance implies the introduction of the factor 3. So, too, the nth term of this series, being $n-1$ terms in advance of the first, is 2.3^{n-1}. As in the case of the arithmetical progression, the nth term is called the general term; thus, giving to n the value 4, we find the fourth term to be 2.3^3 or 54. In like manner we have for the general series,

$$nth\ term = ar^{n-1} \qquad (I)$$

a result which should be remembered.

Ex. 1. The 5th term of a geometrical progression is 162 and the 8th term is 4374; find the progression and the 10th term.

Let a, r denote the first term and the common ratio.

$$\therefore \quad \text{The 5th term} = ar^4$$

$$\therefore \quad ar^4 = 162.$$

Similarly, constructing the 8th term, we have

$$ar^7 = 4374.$$

Therefore, by division

$$r^3 = \frac{4374}{162} = 27$$

$$\therefore \quad r = 3$$

if we regard only the arithmetical cube root, or, in other words, if we consider only real numbers.

$$\therefore \quad a = \frac{162}{3^4} = 2.$$

The progression is then

$$2+6+18+54+162+\ldots\ldots$$

and the 10th term $= 2.3^9$

$= 39366.$

Ex. 2. The nth term of a series is 5.3^n; shew that the series is a geometrical progression and indicate the series.

Whatever be the value of n,

$$\text{The } n\text{th term} = 5.3^n$$

$$\therefore \quad \text{The } (n-1)\text{th term} = 5.3^{n-1}.$$

Therefore the ratio of the nth to the $(n-1)$th term is

$$\frac{5.3^n}{5.3^{n-1}}, \text{ or } 3.$$

Now n is *any* integer, so that the ratio of *any* and therefore of *every* two consecutive terms is 3, and the series is a geometrical progression.

In 5.3^n put $n=1$ and we find that the *first* term is 15. The series is therefore

$$15+45+135+405+\ldots\ldots$$

EXERCISES

(NOTE: In the exercises, G.P. will be employed as an abbreviation for geometrical progression.)

1. Find the 9th and the nth term of each of the following series:

(1) $1+5+25+\ldots\ldots$

(2) $7+14+28+\ldots\ldots$

(3) $2+\frac{2}{3}+\frac{2}{9}+\ldots\ldots$

(4) $1+\frac{1}{2}+\frac{1}{4}+\ldots\ldots$

(5) $1-\frac{1}{2}+\frac{1}{4}-\ldots\ldots$

In each case after finding the nth term test the result by finding from it the first four terms.

2. The 11th term of a G.P. is $\frac{15}{32}$, and the 5th term is 30; find the series, and write down its 17th term.

3. Find three numbers in G.P. such that the third exceeds the sum of the other two by the first while the third exceeds the first by 36.

4. Shew that the series formed by taking every third term of the G.P.

$$7+21+63+\ldots\ldots$$

is also a G.P.

5. Shew that the series formed by taking every 5th term of the G.P.

$$a+ar+ar^2+\ldots\ldots$$

is also a G.P.

6. If each term of a G.P. be multiplied by the same number, the resulting series is a G.P.

7. If corresponding terms of two G.P. be multiplied together, and the results be written in order as terms, shew that the series is a G.P.

Illustrate this theorem.

8. Write down any 7 numbers in G.P. and shew that the average of the 1st and 7th is *not* the same as the average of the 2nd and 6th, or of the 3rd and 5th; but shew that the product of the 1st and 7th is equal to that of the 2nd and 6th, or that of the 3rd and 5th, and is equal to the square of the 4th.

9. The first of 17 terms of a G.P. is $\frac{1}{192}$ and the last is $341\frac{1}{3}$; find the middle term.

10. The nth term of a series ab^{2n}, whatever be the value of n; shew that the series is a G.P.

2. **Geometrical Means.** When three numbers are in geometrical progression, the middle number is called the geometrical mean of the other two. We have then the problem:

To find the geometrical mean of two given numbers.

Let a and b be the given numbers and let x be the mean sought. Then a, x, b are in geometrical progression and by definition

$$\frac{x}{a}=\frac{b}{x}$$

$$\therefore\ x^2=ab$$

$$\therefore\ x=\sqrt{ab}.$$

Thus, the geometrical mean of two numbers is equal to the square root of their product.

As it is supposed that we are dealing with *real* numbers the two numbers whose mean is sought are supposed to be of the same sign, and in extracting the square root of their product we take the sign so as to place the mean between the given numbers.

It is to be noted that, if three or more numbers are in geometrical progression, the numbers are in continued proportion, and that the geometrical mean of two numbers is the mean proportional of those numbers.

In like manner we have the problem of inserting any number of geometrical means between two given numbers.

Ex. Insert 5 geometrical means between $\frac{81}{64}$ and $\frac{1}{9}$. Here $\frac{81}{64}$, the 5 means sought, and $\frac{1}{9}$ make up 7 numbers in geometrical progression. Let r be the common ratio.

$$\therefore \text{ The 7th term} = \frac{81}{64} \cdot r^6$$

$$\therefore \frac{81}{64} \cdot r^6 = \frac{1}{9}$$

$$\therefore r^6 = \frac{64}{81 \times 9}$$

$$\therefore r = \frac{2}{3}$$

taking the positive real sixth root for reasons given. Therefore the means are

$$\frac{27}{32}, \frac{9}{16}, \frac{3}{8}, \frac{1}{4}, \frac{1}{6}.$$

We proceed in the same way in the case of the general problem:

To insert n geometrical means between a and b.

Here a, the n means sought, and b make up $n+2$ terms in geometrical progression. Let r be the common ratio

$$\therefore \text{ The } (n+2)\text{nd term} = ar^{n+1}$$

$$\therefore ar^{n+1} = b$$

$$\therefore r^{n+1} = \frac{b}{a}$$

$$\therefore r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

and the n means may be constructed.

EXERCISES

1. Insert 3 geometrical means between 7 and 567.

2. Insert 5 geometrical means between 5 and 588245.

3. Between 4 and 186624 a certain number of geometrical means have been inserted; the 3rd is 864 and the 5th is 31104. Find the means.

4. Whatever odd number of geometrical means be inserted between a and b, shew that the middle mean is the geometrical mean of a and b.

3. **The Summation Formula.** The sum of any number of terms of a geometrical series may be found without actual addition.

Ex. Find the sum of 8 terms of the series

$$7+21+63+\ldots\ldots$$

Let s denote the sum sought; then, since the common ratio is seen to be 3,

$$s=7+21+63+189+567+1701+5103+15309.$$
$$\therefore \quad 3s=\quad 21+63+189+567+1701+5103+15309+45927.$$

The second equation is formed from the first by multiplying each term by 3 and setting the result one place to the right. Then, since each term of the given series is formed from the preceding term by multiplying it by 3, we see *in advance* that each number in the series for $2s$ will be below an equal number. Then subtracting the numbers in the first equation from those in the second we have

$$2s=45927-7=45920$$
$$\therefore \quad s=22960.$$

Consider now the general problem:

To find the sum of n terms of the geometrical progression

$$a+ar+ar^2+\ldots\ldots$$

Here the common ratio is r, so that the nth term is ar^{n-1}. Let s denote the sum sought. Then, having in mind the terms not written, we write

$$s=a+ar+ar^2+\ldots\ldots+ar^{n-3}+ar^{n-2}+ar^{n-1}$$
$$\therefore \quad rs=\quad ar+ar^2+\ldots\ldots+ar^{n-3}+ar^{n-2}+ar^{n-1}+ar^n.$$

Then by subtraction we have

$$s-rs=a-ar^n$$
$$\therefore \quad s(1-r)=a(1-r^n)$$

$$\therefore \quad \left.\begin{aligned} s&=a\frac{1-r^n}{1-r}, \\ \text{or, which is the same thing,} \quad & \\ s&=a\frac{r^n-1}{r-1}. \end{aligned}\right\} \qquad (II)$$

These results should be carried in memory. As stated, they are equivalent, but it is more natural to employ the former when $r<1$, the latter when $r>1$.

Cor. From the results in (II) we find at once

$$\frac{1-r^n}{1-r}=\frac{r^n-1}{r-1}=1+r+r^2+\ \ldots\ldots\ +r^{n-1}$$

which may be regarded as affording a proof of the theorem that $1-r^n$, for all integral values of n, is divisible by $1-r$, the quotient being

$$1+r+\ \ldots\ldots\ +r^{n-1}.$$

EXERCISES

1. Sum to 29 terms, after the manner of the general theorem (*i.e.*, not quoting the formula), each of the following series :

(1) $2+10+50+\ldots\ldots$

(2) $1+\frac{1}{7}+\frac{1}{49}+\ldots\ldots$

(3) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots\ldots$

(4) $\frac{2}{3}-\frac{4}{9}+\frac{8}{27}-\frac{16}{81}+\ldots\ldots$

(5) $a+a^3+a^5+a^7+\ldots\ldots$

2. Sum to n terms, after the manner of the general theorem, each of the following series :

(1) $3+12+48+\ldots\ldots$

(2) $6+3+1\frac{1}{2}+\ldots\ldots$

(3) $1-\frac{1}{3}+\frac{1}{9}-\ldots\ldots$

In each case *test* the result by assigning to n the value 4.

3. Sum to n terms,

$$x^{n-1}+x^{n-2}y+x^{n-3}y^2+\ldots\ldots$$

and point out the significance of the result.

4. In the general G.P. of n terms

$$a+ar+ar^2+\ldots\ldots+ar^{n-1}$$

shew that the product of the rth term from the beginning and the rth term from the end is equal to the product of the first and the last term.

5. Sum to n terms :

(1) $\left(1+\frac{1}{x}\right)^2+\left(1+\frac{1}{x^2}\right)^2+\left(1+\frac{1}{x^3}\right)^2+\dots\dots$

(2) $(a+b)^2+(a^2+b^2)^2+(a^3+b^3)^2+\dots\dots$

6. The product of three numbers in G.P. is 1728 and the sum of the first and the last is 25 ; find the numbers.

7. Divide the number 221 into three parts in geometrical progression such that the third exceeds the first by 136.

8. The sum of the first three terms of a G.P. is 228 and the sum of the first six terms 997·5 ; find the series.

9. Shew that if, in a G.P., each term be subtracted from the term following it, the successive differences form a G.P.

10. The sum of n terms of a certain series is $h(r^n-1)$; shew that the series is a G.P.

11. If the pth, qth, rth terms of a G.P. are a, b, c respectively, shew that

$$a^{q-r}.\ b^{r-p}.\ c^{p-q}=1.$$

12. Sum to n terms

$$9+99+999+\dots\dots$$

13. Sum to n terms

$$(a+b)+(a^2+ab+b^2)+(a^3+a^2b+ab^2+b^3)+\dots\dots$$

4. **The Infinite Geometrical Series.** In the case of the geometrical series

$$1+2+2^2+\dots\dots$$

the nth term is 2^{n-1} and the sum of n terms is given by the formula

$$s_n=\frac{2^n-1}{2-1}=2^n-1,$$

where the subscript n in s_n indicates that it is a question of the sum of n terms. If to n be given the value 25 we find that the 25th term is 16,777,216 and that the sum of 25 terms is 33,554,431 and the rapidity with which the terms and the sums of terms increase with increase of n is very striking. It is easily seen that n may be taken sufficiently large to make either the nth term or the sum of n terms greater than any assigned number however large.

Consider next the geometrical series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots\ldots$$

Here the nth term, denoted by u_n, is

$$u_n = \frac{1}{2^{n-1}}$$

and the sum of n terms is

$$s_n = \frac{1 - \dfrac{1}{2^n}}{1 - \dfrac{1}{2}} = 2 - \frac{1}{2^{n-1}}.$$

If to n be given the value 25, we find that

$$u_{25} = \frac{1}{16,\ 777,\ 216},\quad s_{25} = 2 - \frac{1}{16,\ 777,\ 216},$$

and are struck by the smallness of the 25th term and by the proximity to 2 of the sum of 25 terms. From the expressions for u_n and s_n it is plain that, with increase of n, the value of u_n becomes smaller and smaller, and that the sum becomes more and more nearly equal to 2. It is plain too that n may be taken sufficiently large to make u_n less than any assigned positive quantity however small, and to make the sum, which is less than 2, differ from 2 by less than any assigned positive number, however small. Limiting the attention to s_n, *i.e.*, to the sum of n terms, we see that

(1) s_n is *always* less than 2;

(2) s_n increases with n;

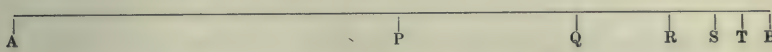
(3) n may be taken sufficiently large to make s_n approach 2 more nearly than by any assigned positive quantity however small, so that there is no number less than 2 that s_n cannot be made to exceed.

We say then that 2 is the **limit to the sum of n terms** as n is indefinitely increased, or, more briefly, that the **infinite series**

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \ldots\ldots$$

has 2 as its sum. The series may be looked upon as a perfectly definite though not the simplest way of giving the number 2.

That s_n may, by increasing n, be made to differ from 2 by as small a quantity as we please is rendered even more striking by representing the terms taken by lengths measured on a straight line.



AB measures 2; AP, 1; PQ, $\frac{1}{2}$; QR, $\frac{1}{4}$; RS, $\frac{1}{8}$; ST, $\frac{1}{16}$; etc. Then AQ measures the sum of 2 terms; AR of 3 terms; AS of 4 terms; AT of 5 terms; etc. Not many terms need be taken to make the sum practically 2, though the sum 2 is never reached.

The general series

$$a + ar + ar^2 + \ldots\ldots$$

may now be treated more concisely.

(I). Suppose r numerically greater than 1 or, in symbols, $|r| > 1$. Then, denoting by s_n the sum of n terms, we have

$$s_n = a\frac{r^n - 1}{r - 1} = a \cdot \frac{r^n}{r - 1} - \frac{a}{r - 1}.$$

Now, r being numerically greater than 1, r^n increases in numerical value as n increases, and n may be taken sufficiently large to make r^n numerically greater than any assigned number however large. Therefore also, a and r being given values, n may be taken large enough to make

$$\frac{ar^n}{1 - r}, \text{ and consequently } \frac{ar^n}{1 - r} - \frac{a}{1 - r}$$

numerically greater than any assigned number. Thus, as n increases indefinitely, the numerical value of s_n tends beyond all limit and we may not in this case speak of the limit of the sum of n terms.

(II). Suppose $|r| < 1$.

Then,

$$s_n = a\frac{1 - r^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

Now, r being numerically less than 1, r^n decreases in numerical value as n increases, and n may be taken sufficiently large to make r^n numerically less than any assigned positive quantity, however small.

Therefore, also, a and r being given values, n may be taken large enough to make

$$\frac{ar^n}{1-r}$$

smaller than any assigned positive quantity, however small. Thus, as n increases indefinitely the numerical value of s_n differs less and less from that of $\frac{a}{1-r}$, and n may be taken large enough to make s_n differ from $\frac{a}{1-r}$ by less than any assigned positive number, however small.

We say then that:

The limit of the sum of n terms of the series

$$a + ar + ar^2 + \ldots\ldots \qquad (|r| < 1).$$

as n increases indefinitely is $\frac{a}{1-r}$,

or, in other words,

The sum of the infinite series

$$a + ar + ar^2 + \ldots\ldots \qquad (|r| < 1).$$

is $\frac{a}{1-r}$.

Ex. Find the value of $0 \cdot \dot{5}$.

$$0 \cdot \dot{5} = 0 \cdot 55555 \ldots\ldots \text{ in infinitum.}$$

$$= \frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \ldots\ldots \text{ in infinitum.}$$

This is an infinite G. P. whose common ratio is $\frac{1}{10}$, which is less than 1,

$$\therefore \quad 0 \cdot \dot{5} = \frac{\frac{5}{10}}{1 - \frac{1}{10}} = \frac{5}{9}.$$

EXERCISES

1. Write down the expression for the sum of n terms of each of the following series, and find the limit to the sum as n is indefinitely increased :

(1) $1+\frac{1}{3}+\frac{1}{9}+\ldots\ldots$

(2) $1+\frac{2}{3}+\frac{4}{9}+\ldots\ldots$

(3) $1-\frac{1}{2}+\frac{1}{4}-\ldots\ldots$

2. Write down the sums of the following infinite series :

(1) $5+\frac{10}{3}+\frac{20}{9}+\ldots\ldots$

(2) $2+\frac{6}{7}+\frac{18}{49}+\ldots\ldots$

(3) $1+\frac{1}{1{\cdot}05}+\frac{1}{(1{\cdot}05)^2}+\ldots\ldots$

3. Shew that any term of the infinite series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots$$

is equal to the sum of the infinite series of succeeding terms.

4. Shew that any term of the infinite series

$$1+\frac{1}{3}+\frac{1}{9}+\ldots\ldots$$

is twice the sum of the infinite series of succeeding terms.

5. Construct the G.P. whose first term is 1 and in which any term is three-fourths the sum of the infinite series of succeeding terms.

6. Find the value of each of the following recurring decimals :

$$0{\cdot}\dot{7},\ 0{\cdot}\dot{0}1\dot{3},\ 0{\cdot}73\dot{2}1\dot{9}.$$

7. The middle points of the sides of a square are joined to form another square ; the middle points of this square are joined to form a new square ; and so on indefinitely. Find the sum of all the squares thus formed.

8. Write down the sum of the infinite series

$$1+\frac{3}{4}+\frac{9}{16}+\ldots\ldots$$

Find also the sum of the infinite series which follows the nth term of this series and hence find the error made in taking the first nine terms as the equivalent of the infinite series.

Obtain a simple rough estimate of this error.

9. Find the product of the sums of the two infinite series

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots\ldots$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots$$

and shew that it is equal to the sum of the infinite series

$$1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots\ldots$$

10. The middle points of the sides of any given triangle are joined to form a new triangle; the middle points of this triangle are joined to form a new triangle; and so on indefinitely. Find the sum of all the triangles thus formed in terms of A, the area of the given triangle.

Find also the sum of all the triangles thus formed that are similarly situated to the original triangle and of those that lie in a reversed sense.

11. In a circle of radius r, a square is inscribed; in this square a circle is inscribed; in this circle a square is inscribed; and so on indefinitely. Find the sum of the areas of the circles, the sum of the areas of the squares, and shew that these sums are to each other as the area of the first circle to that of the first square.

12. Find the non-terminating decimals which are the equivalent of $\frac{4}{9}, \frac{5}{7}, \frac{11}{36}$.

Relating the results to infinite geometrical series, shew that they are the equivalents of the fractions which yielded them.

13. State the reasons why it is not possible to attach the idea of sum to the infinite series

$$1+2+4+8+16+\ldots\ldots$$

14. If s_n denotes the sum of n terms of the infinite series

$$\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\ldots\ldots$$

shew that s_n is less than the sum of n terms of the series

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots$$

and therefore less than 2. Noting that this is true, whatever be n, and that s_n increases as n is taken greater and greater, adduce any reason for thinking that the idea of sum can be attached to the given series.

III

HARMONICAL PROGRESSION

1. **Definition.** A series is said to be an **harmonical progression** when the series formed by the reciprocals of its terms is an arithmetical progression.

Let a, b, c be three numbers in harmonical progression. Then

$$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$$

are in arithmetical progression. Therefore

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$$

$$\therefore \quad \frac{a-b}{ab}=\frac{b-c}{bc}$$

$$\therefore \quad \frac{ab}{bc}=\frac{a-b}{b-c}$$

$$\therefore \quad \frac{a}{c}=\frac{a-b}{b-c}.$$

Thus, *if three numbers are in harmonical progression, the first is to the third as the difference between the first and the second is to the difference between the second and the third.*

This important property of three numbers in harmonical progression is sometimes taken as the definition of an harmonical progression of three terms.

Problems in harmonical progression may be solved by considering the analogous problem in arithmetical progression. This will be illustrated. It is to be remarked that there exists no formula for the sum of n terms of an harmonical progression.

Ex. Shew that 35, 45, 63 are in harmonical progression and find the nth term.

Consider $\frac{1}{35}, \frac{1}{45}, \frac{1}{63}$.

Then $\frac{1}{45}-\frac{1}{35}=-\frac{2}{315}$ and $\frac{1}{63}-\frac{1}{45}=-\frac{2}{315}$, so that $\frac{1}{35}, \frac{1}{45}, \frac{1}{63}$ are in arithmetical progression and therefore 35, 45, 63 in harmonical progression.

Next, the series

$$\frac{1}{35}+\frac{1}{45}+\frac{1}{63}+\cdots\cdots$$

being arithmetical with common difference $\frac{2}{315}$ will have for nth term

$$\frac{1}{35}+(n-1)\left(-\frac{2}{315}\right), \text{ or } \frac{11-2n}{315}.$$

Therefore the harmonical progression

$$35+45+63+\cdots\cdots$$

will have for nth term $\frac{315}{11-2n}$.

EXERCISES

(NOTE: In the exercises, H.P. will be employed as an abbreviation for harmonical progression.)

1. Shew that 15, 21, 35 are in H.P. and continue the series four terms.

2. The first two terms of an H.P. are 2 and 3; find the next two terms.

3. The 5th term of an H.P. is 168 and the 8th term is 108; find the first three terms.

4. The vertical angle C of a triangle ABC is bisected internally and externally by straight lines which meet the base in P and Q respectively; shew that AP, AB, AQ are in H.P.

5. If three numbers are defined to be in harmonical progression when the first is to the third as the difference between the first and the second is to the difference between the second and the third, and if a series is defined to be in harmonical progression when every consecutive three terms are in harmonical progression, shew that the reciprocals of the terms of an H.P. are in A.P.

2. **Harmonical Means.** If three numbers are in harmonical progression the middle number is called the **harmonical mean** of *the other two numbers;* we have then the problem:

To find the harmonical mean of a and b.

Let x be the mean sought. Then a, x, b are in harmonical progression and therefore $\frac{1}{a}$, $\frac{1}{x}$, $\frac{1}{b}$ are in arithmetical progression.

Therefore

$$\frac{1}{x}-\frac{1}{a}=\frac{1}{b}-\frac{1}{x}$$

$$\therefore \quad \frac{2}{x}=\frac{a+b}{ab},$$

$$\therefore \quad x=\frac{2ab}{a+b}.$$

Thus, *the harmonical mean of two numbers is the quotient of twice their product by their sum.*

If a and b are two positive numbers and if their arithmetical, geometrical and harmonical means be denoted by A, G, H, respectively, we may compare the values of A, G and H. For we have

$$A=\frac{a+b}{2};\ G=\sqrt{ab};\ H=\frac{2ab}{a+b}.$$

Then

$$AH=\frac{a+b}{2}\cdot\frac{2ab}{a+b}=ab=G^2$$

so that $G=\sqrt{AH}$ and G is not only the geometrical mean of a and b, but also the geometrical mean of A and H.

Again

$$A-G=\frac{a+b}{2}-\sqrt{ab}$$

$$=\frac{a-2\sqrt{ab}+b}{2}$$

$$=\frac{(\sqrt{a}-\sqrt{b})^2}{2}.$$

But, supposing a and b unequal, $(\sqrt{a}-\sqrt{b})^2$ being the square of a real number is positive, so that $A-G$ is positive or $A>G$. Further since $AH=G^2$ we have $\frac{A}{G}=\frac{G}{H}$, and A being greater than G we have $G>H$, so that $A>G>H$.

If $a=b$ it is readily seen that $A=G=H$.

EXERCISES

1. Insert two harmonical means between 10 and 20.

2. Find the arithmetical, the geometrical, and the harmonical mean of 3 and 12.

3. If the lines AB, BC measure x and y respectively, placing AB, BC in continuous straight line ABC, construct the arithmetical and geometrical means of x and y and shew that the former exceeds the latter unless x and y are equal.

4. If the harmonic mean of two numbers is to their geometric mean as 4 to 5, prove that the quantities are in the ratio of 1 to 4.

5. If the pth term of an H.P. is q and the qth term is p, where p and q are given integers, find the $(p+q)$th term.

6. The pth, qth, rth terms of an H.P. are a, b, c; prove that

$$(q-r)bc+(r-p)ca+(p-q)ab=0.$$

IV

ADDITIONAL IMPORTANT SERIES

In this section will be considered certain series whose sums can be obtained from the results found or by a modification of the methods employed in the preceding sections. We recall (Ex. 3, p. 60) the sum of the first n natural numbers:

$$1+2+3+\ldots\ldots+n=\frac{n(n+1)}{2}.$$

1. **The Squares of the Natural Numbers.** It is proposed to find a formula for the sum of the squares of the first n natural numbers.

Denote the sum by S_n. Then

$$S_n=1^2+2^2+3^2+\ldots\ldots+n^2.$$

We have identically, *i.e.*, for all values of r,

$$r^3-(r-1)^3=3r^2-3r+1;$$

Hence, giving to r in succession the values $n, n-1, n-2, \ldots\ldots 2, 1,$ we have

$$
\begin{array}{llllll}
n^3 & -(n-1)^3 = & 3n^2 & & -3n & +1, \\
(n-1)^3 & -(n-2)^3 = & 3(n-1)^2 & - & 3(n-1) & +1, \\
(n-2)^3 & -(n-3)^3 = & 3(n-2)^2 & - & 3(n-2) & +1, \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots & \ldots \\
3^3 & -\quad 2^3 = & 3.3^2 & - & 3.3 & +1, \\
2^3 & -\quad 1^3 = & 3.2^2 & - & 3.2 & +1, \\
1^3 & -\quad 0^3 = & 3.1^2 & - & 3.1 & +1.
\end{array}
$$

Then, having regard to the lines not written but merely indicated by the dotted line, and noting that there are in all n lines, we have by addition

$$n^3 - 0^3 = 3(1^2 + 2^2 + 3^2 + \ldots\ldots + n^2) - 3(1 + 2 + 3 + \ldots\ldots + n) + n.$$

$$\therefore \quad n^3 = 3S_n - 3.\ \frac{n(n+1)}{2} + n$$

$$\therefore \quad 3S_n = n^3 + 3.\ \frac{n(n+1)}{2} - n$$

$$= \frac{n}{2}(2n^2 + 3n + 1)$$

$$= \frac{n(n+1)(2n+1)}{2}$$

$$\therefore \quad S_n = \frac{n(n+1)(2n+1)}{6}.$$

This result should be retained in memory.

Ex. Sum to n terms

$$1.3 + 2.5 + 3.7 + \ldots\ldots$$

Each term is the product of two factors. The *first* factors are 1, 2, 3, ... so that the nth *first* factor is n. The *second* factors are 3, 5, 7, ... so that the nth *second* factor is $3+(n-1)2$ or $2n+1$. The nth term of the series is therefore $n(2n+1)$, which equals $2n^2+n$. Now give to n the values 1, 2, 3, ... n, and we have, denoting the sum sought by S_n,

$$S_n=(2.1^2+1)+(2.2^2+2)+(2.3^2+3)+\ldots\ldots+(2.n^2+n)$$

$$=\begin{cases} 2(1^2+2^2+\ldots\ldots+n^2) \\ +(1+2+\ldots\ldots+n) \end{cases}$$

$$=2.\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)(2n+1)}{3}+\frac{n(n+1)}{2},$$

a formula which will give the sum of any number of terms The result may be brought to the simpler form

$$\frac{n(n+1)(4n+5)}{6}.$$

EXERCISES

Assuming the formulæ for the sum of the first n natural numbers and for the sum of their squares, find the sum to n terms of each of the following series, *testing* the result by assigning to n the values 1, 2, 3 :

1. $1^2+3^2+5^2+7^2+\ldots\ldots$
2. $1.2+2.3+3.4+4.5+\ldots\ldots$
3. $2^2+4^2+6^2+8^2+\ldots\ldots$
4. $3.7+5.10+7.13+9.16+\ldots\ldots$
5. $a^2+(a+b)^2+(a+2b)^2+(a+3b)^2+\ldots\ldots$

2. **The Cubes of the Natural Numbers.** It is proposed to find the sum of n terms of the series

$$1^3+2^3+3^3+\ldots\ldots$$

It is readily seen that the series is neither arithmetical nor geometrical, so that it calls for a special method of treatment.

Denote the sum sought by S_n so that

$$S_n=1^3+2^3+3^3+\ldots\ldots+n^3.$$

We have identically

$$r^4-(r-1)^4=4r^3-6r^2+4r-1.$$

Therefore, giving to r the values $n, n-1, n-2, \ldots\ldots 2, 1$, we have

$$\begin{aligned}
n^4 - (n-1)^4 &= 4n^3 - 6n^2 + 4n - 1,\\
(n-1)^4 - (n-2)^4 &= 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1,\\
(n-2)^4 - (n-3)^4 &= 4(n-2)^3 - 6(n-2)^2 + 4(n-2) - 1,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
2^4 - 1^4 &= 4.2^3 - 6.2^2 + 4.2 - 1,\\
1^4 - 0^4 &= 4.1^3 - 6.1^2 + 4.1 - 1.
\end{aligned}$$

Therefore, by addition, we have, since there are n lines,

$$\begin{aligned}
n^4 - 0^4 = 4\,(1^3 + 2^3 + \ldots\ldots + n^3) &- 6\,(1^2 + 2^2 + \ldots\ldots + n^2)\\
&+ 4\,(1 + 2 + \ldots\ldots + n) - n
\end{aligned}$$

$$\therefore \quad n^4 = 4\,S_n - 6.\frac{n\,(n+1)\,(2n+1)}{6} + 4.\frac{n\,(n+1)}{2} - n$$

$$\begin{aligned}
\therefore\ 4\,S_n &= n^4 + n - n\,(n+1)\,(2n+1) - 2n\,(n+1)\\
&= n\,(n+1)\left\{n^2 - n + 1 + 2n + 1 - 2\right\}\\
&= \left\{n\,(n+1)\right\}^2
\end{aligned}$$

$$\therefore \quad S_n = \left\{\frac{n\,(n+1)}{2}\right\}^2.$$

Cor. $1^3 + 2^3 + \ldots\ldots + n^3 = (1 + 2 + \ldots\ldots + n)^2$.

Ex. Sum to n terms the series

$$1.3.5 + 3.5.7 + 5.7.9 + \ldots\ldots$$

As in the preceding article it is seen that the nth term is

$$(2n-1)\,(2n+1)\,(2n+3)$$

which is equal to $8n^3 + 12n^2 - 2n^2 - 3$.

Give to n the values $1, 2, 3, \ldots\ldots, n$, and, denoting the sum sought by S_n,

we have

$$S_n=(8.1^3+12.1^2-2.1-3)+(8.2^3+12.2^2-2.2-3)+\ldots\ldots$$
$$+(8.n^3+12.n^2-2.n-3).$$

$$=\left\{\begin{array}{l} 8(1^3+2^3+\ldots\ldots+n^3) \\ +12(1^2+2^2+\ldots\ldots+n^2) \\ -\ 2(1+2+\ldots\ldots+n) \\ -\ \ (3+3+\ldots\ldots+3) \end{array}\right\}$$

$$=8\left\{\frac{n(n+1)}{2}\right\}^2+12\frac{n(n+1)(2n+1)}{6}-2.\frac{n(n+1)}{2}-3n$$

$$=2n^2(n+1)^2+2n(n+1)(2n+1)-n(n+1)-3n$$

which may be reduced to the simpler form

$$n(2n^3+8n^2+7n-2).$$

EXERCISES

Sum to n terms each of the following series, testing the results by giving to n the values 1, 2, 3:

(1) $1^3+3^3+5^3+7^3+\ldots\ldots$

(2) $1.2.3+2.3.4+3.4.5+\ldots\ldots$

(3) $2^3+4^3+6^3+8^3+\ldots\ldots$

(4) $1.3.5+2.5.8+3.7.10+\ldots\ldots$

(5) $(a+b)^3+(a+2b)^3+(a+3b)^3+\ldots\ldots$

3. The Arithmetico-Geometric Series. It is proposed to find the sum of n terms of the series

$$a+(a+b)r+(a+2b)r^2+\ \ldots\ldots$$

where each *term* is formed by multiplying corresponding terms of the arithmetical series

$$a+(a+b)+(a+2b)+\ \ldots\ldots$$

and the geometrical series

$$1+r+r^2+\ \ldots.$$

The nth term is seen to be $(a+\overline{n-1}b)r^{n-1}$. Denote the sum by S_n so that

$$S_n=a+(a+b)r+(a+2b)r^2+\ldots\ldots+(a+\overline{n-1}b)r^{n-1}$$

$$\therefore\ r.S_n= \qquad ar+\ (a+b)r^2+\ldots\ldots+(a+\overline{n-2}b)r^{n-1}+(a+\overline{n-1}b)r^n.$$

Then, by subtraction,

$$S_n(1-r)=a+(br+br^2+\ \ldots\ldots\ +br^{n-1})-(a+\overline{n-1}b)r^n$$

$$=a+br.\frac{1-r^{n-1}}{1-r}-(a+\overline{n-1}b)r^n,$$

since the series within the brackets is geometrical and consists of $n-1$ terms.

$$\therefore\quad S_n=\frac{a}{1-r}-\frac{(a+\overline{n-1}b)r^n}{1-r}+\frac{br(1-r^{n-1})}{(1-r)^2}.$$

Here the method is the important thing and the result need not be retained in memory.

EXERCISES

1. Sum to n terms

$$1+2x+3x^2+4x^3+\ldots\ldots$$

2. Sum to n terms

$$1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots\ldots$$

testing the result by giving to n the values 1, 2, 3.

3. Sum to n terms

$$7+12x+17x^2+22x^3+\ldots\ldots$$

4. Sum to n terms

$$1-2x+3x^2-4x^3+\ldots\ldots$$

EXAMPLES

1. Continue each of the following series three terms:

(1) 48, 60, 72;

(2) 48, 60, 75;

(3) 48, 60, 80.

2. The arithmetical mean of two numbers is 64 and their harmonical mean is 60; find the numbers.

3. The sum of four numbers in A.P. is 72 and the product of the extremes is to the product of the means as 27 to 35; find the numbers.

4. If the arithmetical mean between a and b is twice as great as the geometrical mean, shew that $a : b :: 2 + \sqrt{3} : 2 - \sqrt{3}$.

Obtain this result also geometrically.

5. If a, b, c are three given numbers, find the numbers which if added to each of them will give sums (1) in A.P., (2) in G.P., (3) in H.P.

6. If $2n+1$ terms of the series 1, 3, 5, 7, 9, ... be taken, shew that the sum of the alternate terms 1, 5, 9, ... will be to the sum of the remaining terms as $n+1$ to n.

7. On the ground lie n stones at intervals of 5 yards; how far will a person at the first stone have to travel to go and bring them one by one to the first stone?

8. On the ground lie n stones at intervals of 1 yard, 3 yards, 5 yards, 7 yards, etc.; how far will a person at the first stone have to travel to go and bring them one by one to the first stone?

9. Sum to n terms:

(1) $0.9 + 0.99 + 0.999 + \ldots\ldots$ sum the T_n term

(2) $0.7 + 0.77 + 0.777 + \ldots\ldots$

10. The series of natural numbers is divided into groups as follows:

$$1;\ 2,\ 3;\ 4,\ 5,\ 6;\ 7,\ 8,\ 9,\ 10;\ \text{etc.}$$

Find the sum of the numbers in the rth group.

11. Between a and b are inserted n geometrical means; find the sum of those means.

12. The sides of a right-angled triangle are in A.P.; shew that they are in the ratio of $3 : 4 : 5$.

13. Sum to n terms:

(1) $1 + (1+b)r + (1+b+b^2)r^2 + \ldots\ldots$

(2) $(x+a) + (x^2+2a) + (x^3+3a) + \ldots\ldots$

(3) $1.1^2 + 2.3^2 + 3.5^2 + \ldots\ldots$

14. Sum to n terms:

$$1^2+2^2x+3^2x^2+4^2x^3+\dots..$$

15. If a, b, c, d are in G.P., shew that

$$(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2.$$

Prove also the converse proposition.

16. If a^2, b^2, c^2 are in A.P., shew that $b+c$, $c+a$, $a+b$ are in H.P.

17. Shew that a, b, c are in A.P., G.P., or H.P., according as

$$\frac{a-b}{b-c}=\frac{a}{a},\ \frac{a}{b},\text{ or }\frac{a}{c}.$$

18. If the sum of n terms of a series is $a+bn+cn^2$ find the rth term and the nature of the series.

19. The sum of n terms of a certain A.P. is $(2n)^2$ for all values of n; find the series.

20. Sum to $2n$ terms

$$\frac{a}{r}-\frac{b}{r^2}+\frac{a}{r^3}-\frac{b}{r^4}+\dots..$$

21. If a, b, c are in H.P., shew that

(1) $a:a-b::a+c:a-c$;

(2) $\dfrac{b+a}{b-a}+\dfrac{b+c}{b-c}=2$;

(3) $\dfrac{1}{c-a}+\dfrac{1}{c-b}=\dfrac{1}{a}+\dfrac{1}{b}$.

22. If

$$x=a+\frac{a}{r}+\frac{a}{r^2}+\dots..\ \textit{in inf.};$$

$$y=b-\frac{b}{r}+\frac{b}{r^2}-\dots..\ \textit{in inf.};$$

$$z=c-\frac{c}{r^2}+\frac{c}{r^4}-\dots..\ \textit{in inf.};$$

shew that $xy:z::ab:c$.

23. If a, b, c are the pth, qth, rth terms of both an A.P. and a G.P., shew that

$$a^{b-c}b^{c-a}c^{a-b}=1.$$

24. If x, a_1, a_2, y are in A.P., x, g_1, g_2, y in G.P., and x, h_1, h_2, y in H.P. then

$$\frac{a_1+a_2}{h_1+h_2}=\frac{g_1g_2}{h_1h_2}; \quad xy=a_1h_2=a_2h_1; \quad a_1h_2+a_2h_1=2g_1g_2.$$

25. If x, a_1, a_2, a_3, y are in A.P., and x, h_1, h_2, h_3, y in H.P. shew that

$$xy=a_1h_3=a_2h_2=a_3h_1.$$

26. If S_n, S_{2n}, S_{3n} are the sums of n terms, $2n$ terms, $3n$ terms of a G.P., shew that

$$S_n(S_{3n}-S_{2n})=(S_{2n}-S_n)^2.$$

CHAPTER VI

ANNUITIES, DEBENTURES, AND SINKING FUNDS

1. Preliminary. The geometrical series finds an important application in those problems of finance which involve a succession of payments,—generally equal payments—made or to be made at the end of certain equal intervals of time. Certain facts, supposed known are here recalled :

(1) Money is regarded always as being capable of investment. This implies that when a sum is supposed to become the property of A, say, it becomes interest bearing to A's advantage.

(2) The rate of interest is given as the number (of dollars, pounds, francs, etc.), which *one hundred* will yield as interest in *one year*, also as the number (of dollars, etc.) which *one* will yield as interest in *one year*. Thus "5 per cent. per annum" or "·05 per annum" is an interest quotation. The former mode under the form "5%," often without reference to the year as this is taken for granted, is the one adopted in ordinary business. In the mathematical theory the rate is usually based on the unit, so that ·05 replaces 5%. The symbol i is in general used to designate the rate, so that i means the interest on 1 for one year.

Often when the rate is quoted with reference to the year, there is an understanding that interest accrues or becomes due at the end of a period other than the year, as for example at the end of each half-year or each quarter-year. Thus "5% interest (per annum) compounded half-yearly" means that interest falls due every half-year, and the rate for the half-year is $2\frac{1}{2}\%$.

(3) If a sum, as \$400, is lent at ·05 for 3 years, and if the interest at the end of each year is not paid but allowed to accumulate, interest as it becomes due bearing interest, the amount necessary to repay the loan at the end of the time is

$$\$400 \times (1{\cdot}05)^3.$$

This sum is called the **amount**, and the **principal**, \$400, will have yielded as **interest**

$$\$400 \times (1{\cdot}05)^3 - \$400.$$

In the general case where the principal is \$A, the rate i and the time n, where n is a positive integer,

$$\text{the amount} = A\,(1+i)^n,$$
$$\text{the interest} = A\,[(1+i)^n - 1].$$

(4) If a sum, as \$400, is to become due at the end of 3 years, and if the rate of interest for investments is ·05, the sum which would now discharge the obligation—the equivalent sum, or the **present value**—is

$$\frac{\$400}{(1{\cdot}05)^3},$$

for manifestly this sum, invested now, at the end of three years would amount to \$400.

In the general case the present value of \$A due n years hence, the rate of interest being i, is

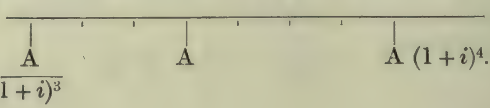
$$\frac{\$A}{(1+i)^n}.$$

The sum that should be allowed off the debt for present payment, namely,

$$\$A\left\{1 - \frac{1}{(1+i)^n}\right\}$$

is called the **discount,** often the **true discount.** It is not to be confounded with the discount employed in finding the present proceeds of notes discounted for short periods, where a *rate of discount* is quoted, whereas in the work now under study everything rests on an accepted *rate of interest.*

(5) From the preceding it is plain, since the present may be any point in time, that $(1+i)^n$ is a factor which will carry a sum forward n years, *i.e.*, will give the equivalent of it n years hence, and that $\dfrac{1}{(1+i)^n}$ is a factor which will carry a sum backward n years, *i.e.*, will give the equivalent sum n years earlier. The diagram indicates three sums which are equivalent, because of a different location in time:



If the interest-period is not one year, which is the period implied in the quoted rate, the adjustment is not difficult. Thus the present value of \$400 due 4 years hence, the rate being ·06 (per annum) compounded half-yearly is

$$\frac{\$400}{(1{\cdot}03)^8}.$$

(6) If a time less than the period for interest computation presents itself in a problem, the common practice is to regard the interest as proportionate to the time. For example, if the sum is \$300, the interest-period one year and the rate 4%, the interest for 57 days is

$$\frac{57}{365} \text{ of } (\$300 \times 0{\cdot}04).$$

EXERCISES

1. Write down the expression for the amount of
   (1) \$700 in 5 years at 6% ;
   (2) \$250 in 6 years at 7% ;
   (3) \$400 in 7 years at 5%, compounded half-yearly ;
   (4) \$375 in 4 years at $3\frac{1}{2}$%, compounded quarterly ;
   (5) \$200 in $3\frac{1}{2}$ years at 4%, compounded quarterly.

2. Write down the expression for the present value of
   (1) \$200 due 3 years hence, the rate of interest being 5% ;
   (2) \$700 due 5 years hence, the rate of interest being 6% ;
   (3) \$250 due 3 years hence, the rate of interest being 5%, payable half-yearly ;
   (4) \$400 due 2 years hence, the rate of interest being 5%, payable quarterly ;
   (5) \$960 due $2\frac{1}{2}$ years hence, the rate of interest being 5%, payable quarterly.

3. By actual multiplication, employing the contracted method, compute a working value—say to *five* places of decimals—of each of the following :

$$(1{\cdot}03)^6,\ (1{\cdot}025)^7,\ (1{\cdot}0125)^8,\ (1{\cdot}035)^{14},$$

$$\frac{1}{(1{\cdot}03^6)},\ \frac{1}{(1{\cdot}025)^7},\ \frac{1}{(1{\cdot}0125)^8},\ \frac{1}{(1{\cdot}035)^{14}}.$$

4. Write down an expression for the amounts of

(1) \$450 in 3 years 4 months, the rate of interest being 5% ;

(2) \$720 in 5 years 8 months, the rate of interest being 4% compounded half-yearly.

(3) \$1050 in 4 years 37 days, the rate of interest being 6% compounded half-yearly.

5. The sum of \$800 becomes due at a certain date. If the rate of interest is $3\frac{1}{2}$% compounded quarterly, write down the expression for the sum that would meet the obligation

(1) 2 years 6 months earlier ;

(2) 1 year 3 months earlier ;

(3) 3 months earlier ;

(4) 3 months later ;

(5) 3 years 9 months later.

6. Find the equivalent rate per annum of (i) 6% compounded half-yearly ; (ii) 6% payable quarterly.

2. **Annuities.** Suppose A under obligation to pay B \$800, at the end of each year for 5 years, the first payment to be made one year hence. This set of payments is called an **annuity** of \$800, **beginning now and running for 5 years.** If a rate of interest is agreed upon, say 4%, it is easy to imagine circumstances that would raise the question of the present value of the annuity. For example, B may be a parent, providing for his child, and A a trust company that can undertake to make the payments. There presents itself the problem to find the sum B must give A to have A undertake these annual payments, the rate of interest being as stated. Plainly the present value of the annuity is

$$\frac{\$800}{1\cdot04}+\frac{\$800}{(1\cdot04)^2}+\frac{\$800}{(1\cdot04)^3}+\frac{\$800}{(1\cdot04)^4}+\frac{\$800}{(1\cdot04)^5},$$

the items corresponding to the successive payments. These five terms are in geometrical progression and their sum is

$$\frac{\$800}{1\cdot04}\cdot\frac{1-\dfrac{1}{(1\cdot04)^5}}{1-\dfrac{1}{(1\cdot04)}}\quad\text{or}\quad\frac{\$800}{\cdot04}\cdot\left\{1-\frac{1}{(1\cdot04)^5}\right\}.$$

The general problem is *to find the present value of an annuity* of \$A *beginning now and running n years, the rate being i.*

Noting that the first payment is to be made one year hence, we see that the present values of the successive payments yield the series

$$\frac{\$A}{1+i}+\frac{\$A}{(1+i)^2}+\ldots+\frac{\$A}{(1+i)^n},$$

a geometrical progression, with common ratio $\frac{1}{1+i}$, which is less than unity. The sum of the series is

$$\frac{\$A}{1+i}\cdot\frac{1-\frac{1}{(1+i)^n}}{1-\frac{1}{1+i}}\quad\text{or}\quad\frac{\$A}{i}\cdot\left\{1-\frac{1}{(1+i)^n}\right\}.$$

Should the first payment of an annuity be made 5 years hence it is said to be *deferred* 4 years, as 4 years hence would be the "present" relatively to the annuity. We have then the general problem: *To find the present value of an annuity of* \$A, *deferred* **m** *years and running* **n** *years, the rate of interest being i.*

The first payment being made $m+1$ year hence, the present values of the successive payments yield the series

$$\frac{\$A}{(1+i)^{m+1}}+\frac{\$A}{(1+i)^{m+2}}+\ldots+\frac{\$A}{(1+i)^{m+n}},$$

a geometrical progression of n terms, common ratio $\frac{1}{1+i}$. The sum is

$$\frac{\$A}{(1+i)^{m+1}}\cdot\frac{1-\frac{1}{(1+i)^n}}{1-\frac{1}{1+i}}=\frac{\$A}{i(1+i)^m}\cdot\left\{1-\frac{1}{(1+i)^n}\right\}.$$

If the annuity is to continue to be paid for all time, as when a government agrees to pay for all time to an institution a stated sum each year, the annuity is called a **perpetuity.** The sum and rate being as in the general problems the present values of successive payments yield the infinite series

$$\frac{\$A}{1+i}+\frac{\$A}{(1+i)^2}+\frac{\$A}{(1+i)^3}+\ldots \textit{in inf.}$$

if the first payment to be made one year hence. This is a geometrical series with common ratio less than unity and its sum is

$$\frac{\$A}{1+i} \cdot \frac{1}{1-\frac{1}{1+i}}, \quad \text{or} \quad \frac{\$A}{i},$$

which is therefore the present value of a perpetuity of \$A a year, the first payment being made one year hence, and the rate of interest being i.

If the first payment were to be made $m+1$ years hence, the present value is easily found to be

$$\frac{A}{i} \cdot \frac{1}{(1+i)^m}.$$

Note that:

(i) The present value of a perpetuity can be obtained without reference to the series, from the consideration that if the sum of $\frac{\$A}{i}$ were *surrendered* to a government, the government could equitably undertake to pay therefor the sum \$A at the end of each year for all time, as this would be the yearly interest on $\frac{\$A}{i}$ at rate i.

So also for a deferred perpetuity.

(ii) The formula for the deferred annuity may be obtained from the earlier formula, by writing down the value one year before the first payment, and then reducing to the present.

(iii) The present value of an annuity of \$A beginning now and running for n years is easily seen to be the difference between a perpetuity of \$A beginning now and one of \$A beginning n years hence, so that the formula for it could be obtained from the formulæ for perpetuities.

A further problem connected with annuities is the following:—
To find the accumulated value of a series of n annual payments of \$A *each, the last payment having just been made, the rate of interest being* i.

Here the value of the payments is at once seen to be

$$\$A + \$A \times (1+i) + \$A \times (1+i)^2 + \ldots + \$A \times (1+i)^{n-1},$$

the successive payments, commencing with the last, furnishing the terms in order. This is a geometrical series, common ratio $(1+i)$, and its sum is

$$\frac{\$A\,(1+i)^n - 1}{(1+i) - 1}, \text{ or } \frac{\$A}{i}\left\{(1+i)^n - 1\right\}.$$

Should in any case the period for compounding and the interval between payments be not the same, it is not difficult to adapt the reasoning.

EXERCISES

1. Write down the series giving the present value of each of the following, then find the sum and reduce it to a simple form :

(1) An annuity of \$300 beginning now and running for 4 years, the rate of interest being 3 per cent.

(2) An annuity of \$200, deferred 3 years and running for 5 years, the rate being 4 per cent. \$791·53

(3) An annuity of \$1 beginning now and running for 7 years, the rate of interest being $3\frac{1}{2}$ per cent. \$6·11

In each case test the result by applying the general formula.

2. Write down the series giving the present value of each of the following, then find the sum and reduce it to a simple form.

(1) An annuity of \$1, beginning now and running for 5 years, the rate of interest being 4 per cent., payable half-yearly.

(2) An annuity of \$100, deferred 2 years and running for 7 years, the rate of interest being 6 per cent., payable half-yearly.

(3) An annuity of \$600, beginning now and running for 4 years, the rate being 5 per cent., payable quarterly.

Examine whether the results could be written down at once from the general formula.

3. At the end of each year for 7 years an investor deposits \$720 in a savings bank which allows 4 per cent. interest. Write down the series which gives the amount to his credit just after the last payment, and find the sum, reducing it to a simple form. \$5352·70

(State in a general way how the account would appear in the books of the bank).

4. At the end of each year for 5 years an investor deposits \$500 in a savings bank which allows $3\frac{1}{2}$ per cent. interest, payable quarterly. Write down the series giving the amount to his credit just after the last payment, and find the sum, reducing it to simple form.

5. Write down the series giving the accumulated value just after the last payment of each of the following, then find the sum and reduce it to simple form:

(1) An annuity of \$150 running 6 years, the rate of interest being 5%.

(2) An annuity of \$540 running 5 years, the rate of interest being 4%, payable half-yearly.

(3) An annuity of \$1,000 running 7 years, the rate of interest being 5%, payable quarterly.

6. Find an expression for the present value of twelve payments of \$600 each, made at intervals of six months, the first six months hence, if the rate of interest is 4 per cent.

7. Find an expression for the present value of eighteen payments of \$750 each, made at intervals of six months, the first one year hence, the rate of interest being 6 per cent., payable quarterly.

3. **Notation and Tables.** In the formulæ obtained in the preceding section, certain numbers present themselves, for which special symbols have been retained.

(i) The rate of interest for the fundamental period, - ordinarily one year,—is denoted by i, meaning i on 1 for a year.

(ii) For $1 + i$ which gives the amount of 1 in one year no simpler symbol is called for.

(iii) The number $\frac{1}{1+i}$ which gives the present value of 1 due one year hence, is designated by v, so that v^n is the present value of 1 due n years hence.

(iv) The accumulated value of n annual payments of 1, just after the last payment is denoted by $s_{\overline{n}|}$ so that

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

(v) The present value of n annual payments, the first to be made one year hence, is denoted by $a_{\overline{n}|}$, so that

$$a_{\overline{n}|} = \frac{1 - v^n}{i}.$$

Accordingly, in the notation given, the rate of interest being i,

Amount of \$A in n years,	$= \$A(1+i)^n$.
Present value of \$A due n years hence,	$= \$Av^n$.
Accumulated value of n annual payments of \$A at end of n years,	$= \$As_{\overline{n}\rceil}$.
Present value of n annual payments of \$A, the first being made one year hence,	$= \$Aa_{\overline{n}\rceil}$.

A study of the formulæ of interest and annuities shews that, after the expression for the value sought has been written, to obtain the actual value, a computation, often of great length, has yet to be made. To meet this frequent demand, tables for different rates and periods have been constructed, the fundamental sum involved being 1. Thus i being the equivalent of $2\frac{1}{2}\%$, *i.e.*, $i = \cdot 025$, the table gives

$$(1+i)^{21} = 1\cdot 679582,$$

so that the amount of \$73 in 21 years at $2\frac{1}{2}\%$ is $\$73 \times 1\cdot 679582$, and this simple multiplication gives the result sought.

It is plain that if tables for $(1+i)^n$ and v^n are given, the corresponding values of $a_{\overline{n}\rceil}$, and $s_{\overline{n}\rceil}$, *i.e.*, of $\dfrac{1-v^n}{i}$ and $\dfrac{(1+i)^n - 1}{i}$, could be found by a not very tedious computation. But these numbers are so much in use that tables for them are given.

It is to be remarked that the tables are, except for low values of n, only approximate, a sufficient number of decimal places being retained to ensure a degree of accuracy possible in actual payments. If very large sums of money are involved, a larger number of decimals might have to be retained. For $(1+i)^n$, v^n, $a_{\overline{n}|}$, $s_{\overline{n}|}$, tables running to 40 years for all rates likely to be required, are given. Should values for a longer period be sought it is an easy problem to find them with but little computation. Thus for $n = 60$,

$$a_{\overline{60}|} = a_{\overline{40}|} + a_{\overline{20}|} \times v^{40}.$$

v^n, $s_{\overline{n}|}$, $a_{\overline{n}|}$, involve i, and if it is necessary to indicate the rate it can be done thus: $v^n_{(i)}$, $s_{\overline{n}|(\cdot 03)}$, $a_{\overline{n}|(\cdot 05)}$.

EXERCISES

1. Employing the notation explained, write down immediately the results for all the exercises gives on pages 103, 107, and employ the tables to find results.

2. Employ the tables to find $(1+i)^n$ for

(i) $i = \cdot 045$, $n = 60$;

(ii) $i = \cdot 055$, $n = 65$;

(iii) $i = \cdot 03$, $n = 70$;

(iv) $i = \cdot 05$, $n = 100$;

3. Employ the tables to find v^n, $a_{\overline{n}|}$ and $s_{\overline{n}|}$ for

(i) $i = \cdot 035$, $n = 55$;

(ii) $i = \cdot 04$, $n = 80$;

(iii) $i = \cdot 055$, $n = 63$;

(iv) $i = \cdot 05$, $n = 100$;

4. **Debentures.** The ordinary way for a corporation, as of a city, to raise money needed at once for some undertaking, is by an issue of **debentures.** Suppose the issue to be for \$100,000, the rate of interest to be 5%, and the term 20 years. This means that the purchaser of a debenture of \$100 gives to the corporation the purchase-money, whatever that may be, and is to receive for this, at the end of each year for 20 years, \$5, and at the end of 20 years the sum of \$100. If the current rate of interest were 4%, it is plain that an investor would be willing to pay rather more than \$100 for such a debenture, and rather less if the current rate were 6%.

In connection with this issue several questions arise:

(1) What rate should an investor offer for the debenture to make 6% interest on his money, *i.e*, to assure that on all outstanding capital he will be receiving interest at 6%?

(2) What sum will be immediately realised by the corporation?

(3) What steps will the corporation take to meet its obligation?

These questions will now be dealt with.

(1) The rate of purchase is based on a debenture of \$100. The purchaser will receive an annuity of \$5 for 20 years, first payment one

year hence, and at the end of 20 years the sum of \$100. He wishes to make 6% on all his capital that is invested. Therefore he should pay the present value of the 20-year annuity of \$5 together with the present value of the \$100 to be paid at the end of 20 years, and this should be estimated on the rate of interest 6%.

He should pay then

$$\$5 \times a_{\overline{20}|} + \$100 \times v^{20}$$

which from the tables is seen to be

$$\begin{aligned} &\$5 \times 11{\cdot}46992 + \$100 \times 0{\cdot}311805 \\ &= \$57{\cdot}3496 + \$31{\cdot}1805 \\ &= \$88{\cdot}53. \end{aligned}$$

Note how the purchaser receives his capital back, and his interest. The yearly interest at 6% on \$88·53 = \$5·31. He receives each year \$5·00 so that \$0·31 is not paid. The amount of this sum each year at the end of 20 years is

$$\$0{\cdot}31\ (2) \times s_{\overline{20}|} = \$0{\cdot}31\ (2) \times 36{\cdot}78\ (6) = \$11{\cdot}48.$$

Thus when at the end of 20 years he is given \$100 we see it as made up of this sum \$11·48, and the \$88·53 his original capital.

It is plain then that he has received his money back, and in addition the interest involved in the rate 6%.

(2) The corporation, selling its debentures at \$88·53, will realize from the sale

$$\$88{\cdot}53 \times 1000 = \$88{,}530.$$

In issuing the bonds at 5% when, as we have supposed, capital is seeking and finding investment at 6%, the corporation would have had in mind the raising of \$88,530 although nominally \$100,000 appears on the debentures. It should also be stated that debentures are generally sold to investors not by the corporation directly but through the medium of agents or brokers, who take them at a certain price and sell them to investors at a somewhat higher rate.

(3) The corporation, supposed a city corporation, in general, raises in each of the 20 years by taxation, a sum, sufficient to pay the yearly dividend of \$5000 on the issue and in addition a sum which invested will at the end of the 20 years accumulate to \$100,000 which the maturing debentures will call for. This sum, supposing it can be invested at $4\frac{1}{2}\%$, is obtained from the equation,

$$x \times s_{\overline{20}|} = \$100,000$$

$$\text{or } x = \frac{\$100,000}{s_{\overline{20}|}},$$

$$= \frac{\$100,000}{31{\cdot}37142},$$

3187.62

$$= \$3105{\cdot}05.$$

The fund established by the corporation through the investing each year of \$3105·05 at the supposed rate $4\frac{1}{2}\%$, in order to meet the need for paying \$100,000 at the end of twenty years is called a **Sinking Fund.**

Not essentially different from the debenture is the **Term Bond.** The Bond is a promise to pay a stated sum, called the *par value* of the bond, say \$100, on a stated future day, say Dec. 1, 1937. Usually to the bond, or rather as a part of the bond, are attached *coupons*, one for every year or every half-year, over the period of the bond. These coupons are promises to pay on specified dates stated sums calculated at a certain rate per cent. on the face value of the bond. Thus, on the 1917 issue of 20-year Victory Bonds, each \$100 bond carries forty coupons for \$2·75 each, payable successively on June 1 and Dec. 1 of each year from June 1, 1918, to Dec. 1, 1937. This bond is said to carry half-yearly coupons at $5\frac{1}{2}\%$, or to be a $5\frac{1}{2}\%$ bond. The par value \$100 is payable on Dec. 1, 1937.

EXERCISES

1. Find the amount of

   (i) \$450 in 4 years at $4\frac{1}{2}\%$;

   (ii) \$117·53 in 5 years at 5% ;

   (iii) \$260 in 9 years at 5%, payable half-yearly ;

   (iv) \$113 in 11 years at $4\frac{1}{2}\%$, payable half-yearly ;

   (v) \$72 in 9 years at $3\frac{1}{2}\%$, payable quarterly.

2. Find the present value of

(i) \$4000 due 3 years hence, interest at 6°/。;

(ii) \$2500 due 5 years hence, interest at 4°/。;

(iii) \$750 due 4 years hence, interest at 5°/。, payable half-yearly;

(iv) \$800 due 7 years hence, interest at 5°/。, payable quarterly;

(v) \$120 due 3 years hence, interest at 6°/。, payable quarterly.

3. Find the present value of an annuity of

(i) \$125 beginning now and running for 13 years, interest at 5°/。;

(ii) \$560 beginning now and running for 17 years, interest at 4°/。;

(iii) \$89·75 beginning now and running for 20 years, interest at 4½°/。;

(iv) \$100 beginning now and running for 12 years, interest at 3½°/。;

(v) \$720 beginning now and running for 10 years, interest at 5½°/。.

4. Find the accumulated value just after the last payment of an annuity of

(i) \$1200, 17 annual payments, interest 4°/。;

(ii) \$1000, 20 annual payments, interest 5°/。;

(iii) \$1750, 12 annual payments, interest 5½°/。;

(iv) \$600, 15 annual payments, interest 4½°/。;

(v) \$400, 7 annual payments, interest 3½ °/。.

5. Find the present value of 20 semi-annual payments of \$480, the first to be made six months hence,

(i) interest at 5°/。;

(ii) interest at 5°/。, payable half-yearly.

6. Find the accumulated value just after the last payment of 18 semi-annual payments of \$300.

(i) interest at 5°/。;

(ii) interest at 5°/。, payable half-yearly.

7. Noting that, if the rate of interest is 4°/。, semi-annual payments of \$100 are the equivalent of annual payments of \$202, find the present value of

(i) 20 semi-annual payments of \$750, the first to be made six months hence, the rate of interest being 4%.

(ii) 17 semi-annual payments of \$600, the first to be made six months hence, the rate of interest being 4%.

8. The rate of interest being 5%, payable half-yearly, shew that annual payments of \$100 are the equivalent of semi-annual payments of \$49·38(2).

Employ this fact to make the computation by immediately appealing to the tables of

(1) The present value of 17 annual payments of \$640, the first to be made one year hence, the rate of interest being 5%, compounded half-yearly.

(2) The accumulated value of 17 annual payments of \$640 just after the last payment, the rate of interest being 5°/ₒ, compounded half-yearly.

9. Find the present value of an annuity of \$180, deferred 6 years and running for 20 years, the rate of interest being

(i) 5%.

(ii) 5% compounded half-yearly.

10. By appeal to the tables find the length of time, for the different rates of interest given, that it will take for a sum of money "to double itself," *i.e.*, to accumulate interest equal to itself.

Let r denote the *rate per cent.* interest, and t the time; mark the points (r, t) given by corresponding values, and shew that $t=\frac{70}{r}$ gives a graph that gathers in these points approximately.

11. Find the value just after the last payment of 15 annual payments of \$150 each, the rate of interest being

(i) 5%.

(ii) 5%, compounded half-yearly.

12. Find the present value of 18 half-yearly payments of \$360, the first to be made now, the rate of interest being

(i) 4%.

(ii) 4%, payable half-yearly.

13. An investor is to deposit in a savings bank at intervals of six months, the sums \$100, \$150, \$200, ... to ten deposits, the first to be made six months hence. Find the present value, and the value just after the last deposit of these deposits, the rate of interest being

(i) 4%.

(ii) 4%, payable half-yearly.

14. Payments of \$100 each are to be made year after year. Find the equivalent quarterly payments if the rate of interest is

(i) 6% ;

(ii) 6%, payable half-yearly ;

(iii) 6%, payable quarterly.

15. Find the amount of the annuity, beginning now and running for 15 years that can be bought with \$10,000, the rate of interest being

(i) 5% ; (ii) 5%, payable half-yearly ;

(iii) 6% ; (iv) 6%, payable half-yearly.

16. Find the sum that should be paid for an annuity of \$750 to run for 9 years, the first payment to be made 5 years hence, the rate of interest being

(i) 5% ;

(ii) 5%, payable half-yearly.

17. In the illustration, page 85, the division by $s_{\overline{20|}}$ suggests the advisability of tables for $\frac{1}{s_{\overline{n|}}}$ or $s_{\overline{n|}}^{-1}$. A demand also arises through a similar problem for tables of $a_{\overline{n|}}^{-1}$. From the expressions for $a_{\overline{n|}}$ and $s_{\overline{n|}}$ shew that $s_{\overline{n|}}^{-1} = a_{\overline{n|}}^{-1} - i$, so that only one set of tables will be needed.

18. Noting that for the repayment of a loan of 1 for n years at rate i, the two methods :

(a) A payment of $(1+i)^n$ at the end of n years ;

(b) Annual payments of i and at the end of n years a payment of 1, are equivalent, shew that

(1) The payments in (b) have at the time of the last payment the value $1 + i\, s_{\overline{n|}}$, and hence that $s_{\overline{n|}} = \dfrac{(1+i)^n - 1}{i}$;

(2) The payments in (b) have at the time the loan is made the value $v^n + i\, a_{\overline{n|}}$ and hence that $a_{\overline{n|}} = \dfrac{1 - v^n}{i}$.

19. Find the present value of 20 annual payments of \$250 each, the first to be made one year hence, the rate of interest being 6 per cent. compounded quarterly.

(Note that $a_{\overline{40|}} \div a_{\overline{4|}}$ gives an important element in the computation.)

20. Check the accuracy of the values of $a_{\overline{n}|}$ $(\cdot 05)$ from $n=1$ to $n=10$ by the continuous addition of the numbers in the v^n column.

Similarly employ the values of $(1+i)^n$ for $i=\cdot 04$ to check the accuracy of the first ten values given for $s_{\overline{n}|}$.

21. Bonds of a corporation to the amount of $500,000, bearing interest at 6%, to be redeemed at the end of 20 years, are offered at a price to yield 5% to the investor. Find the price, and supposing the corporation to be able to invest its money at $4\frac{1}{2}\%$ to constitute a sinking fund, find the amount to be raised each year to meet all obligations.

Shew also in what manner the investor will receive his interest, and will have return made of his capital.

22. A man borrows $10,000 at 5%, agreeing to repay the loan by ten equal annual payments, covering principal and interest. Find the amount of the payment.

23. Employ the tables to find for how many years a cash payment of $10,000 will provide an annuity of $800, the rate of interest being 5%.

Denoting by n the number of years, write down the equation that n must satisfy, and comment upon its character.

24. An investor buys an n-year bond, coupons at rate j, so as to yield him an interest rate i on his investment. Shew that

$$(v^n + ja_{\overline{n}|})(1+i) - j = v^{n-1} + ja_{\overline{n-1}|}$$

where v^n and $a_{\overline{n}|}$ are taken at rate i.

CHAPTER VII

PERMUTATIONS AND COMBINATIONS

1. Explanatory. From the four letters a, b, c, d, it is plain that all possible *selections* of three are the following:

$$bcd,\ cda,\ dab,\ abc.$$

These are the **combinations** of four letters three at a time, and they are four in number.

Take any one of these combinations, say bcd; then by interchange of the letters b, c, d, the only *arrangements* of three that can be formed of those letters are the following:

$$bcd,\ bdc,\ cdb,\ cbd,\ dbc,\ dcb.$$

Each of the remaining three combinations will give rise to six such arrangements, so that in all there can be formed twenty-four arrangements of three letters. These are the **permutations** of four letters three at a time.

So, too, we speak generally of the combinations and permutations of *n things r at a time.* The things are ordinarily denoted by letters, different letters denoting dissimilar things and like letters like things. When nothing to the contrary is stated, the things will be supposed to be dissimilar. In the theorems and problems to be treated it will always be a question of the *number* of possible permutations or combinations in question.

EXERCISES

1. Determine the number of combinations and permutations of three letters, (1) one at a time, (2) two at a time, (3) three at a time, by actually forming the combinations and permutations.

2. Determine the number of combinations and permutations of four letters, (1) one at a time, (2) two at a time, (3) three at a time, (4) four at a time, by actually forming the combinations and permutations.

3. Given that the number of permutations of 5 things 3 at a time is 60, shew that the number of combinations of 5 things 3 at a time is 10 and that the number of permutations of 5 things 4 at a time is 60×2 or 120.

2. **The Fundamental Theorem of Permutations.** The theorem will first be illustrated by finding the number of permutations of 5 letters a, b, c, d, e, taken 3 at a time.

The number of ways in which three places, in order as shewn in the diagram,

b	d	e

may be filled by 3 of the letters is plainly the number of permutations sought. The first place may be filled by any one of the 5 letters and therefore in 5 ways. Suppose the first place filled in any of the 5 ways; then the second place may be filled by any one of the 4 remaining letters and therefore in 4 ways. Thus each of the 5 possible ways of filling the first place may be associated with each of the 4 possible ways of filling the second, which makes in all 5.4 ways of filling the first two places. Suppose the first two places filled in any way; the third place may then be filled by any one of the 3 remaining letters. As before, each of the 5.4 ways of filling the first two places may be associated with 3 ways of filling the third, which makes in all 5.4.3 ways of filling the three places. Hence, 5.4.3 or 60 is the number of permutations sought.

One way of filling the places is indicated. The student is recommended to write out all the ways of filling the second after b has been put in the first, and then to write out all the ways in each case for filling the third place.

The general proposition is: *To find the number of permutations of n different things r at a time.*

The number of ways of filling r places in order, each place by one thing, is the number of permutations sought.

The first place may be filled by any one of the n things and therefore in n ways. Suppose it filled in *any* one way; the second place may then be filled by any one of the $n-1$ remaining things and therefore in $n-1$ ways. Each of the n ways of filling the first may thus be associated with $n-1$ ways of filling the second which makes

$n(n-1)$ ways of filling the first two places. In like manner each of the $n(n-1)$ ways of filling the first two places may be associated with $n-2$ ways of filling the third which makes $n(n-1)(n-2)$ ways of filling the first three places. The reasoning may evidently be continued. When $r-1$ places have been filled, there remain $n-\overline{r-1}$ or $n-r+1$ things so that the rth place may be filled in $n-r+1$ ways. Thus the r places may be filled in

$$n(n-1)(n-2)\ldots\ldots(n-r+1)$$

ways, and this then is the number of permutations of n things r at a time. The number is frequently denoted by the significant symbol ${}_nP_r$.

Cor. The number of permutations of n things n at a time, i.e., all together is

$$n(n-1)\ldots\ldots 3.2.1.$$

This number is generally denoted by the symbol $\underline{|n}$ or $n!$ which is read n **factorial.**

Since

$$n(n-1)\ldots\ldots(n-r+1)=\frac{n(n-1)\ldots\ldots(n-r+1)(n-r)(n-r-1)\ldots\ldots 2.1}{(n-r)(n-r-1)\ldots\ldots 2.1}$$

we have

$${}_nP_r=n(n-1)\ldots\ldots(n-r+1)=\frac{n!}{(n-r)!}.$$

Ex. How many numbers of three digits, all different, may be formed from the digits 1, 2, 3, 4, 5?

This is readily seen to be the same as the number of permutations of 5 things 3 at a time and is therefore

$$5.4.3 \text{ or } 60.$$

EXERCISES

1. In how many ways may 5 books be arranged on a shelf?

2. Write down the numbers represented by ${}_{17}P_{13}$, ${}_{12}P_7$, ${}_8P_8$ and compute the value in each case.

3. How many words, each of four letters, may be formed from the letters of the word *comrade?*

4. How many numbers, each of 4 digits, may be made from the nine digits, no digit being employed more than once?

In how many of these will the first digit be an odd digit, and in how many of them will the first digit be 6?

5. How many numbers, each of 4 digits, may be made from the nine digits and the figure 0, no digit being employed more than once?

How many of these will end in 0?

6. How many numbers of 4 figures each may be made from the nine digits, if digits may be repeated?

7. A signal is made by running up on a vertical rope one, two, or three flags. How many signals could be made with 7 flags of different colours?

8. How many words each of four letters, beginning and ending with a consonant, may be made from the letters of the word *tambour?*

9. In how many ways may 5 ladies and 5 gentlemen be assigned to 10 seats in a row, no two ladies to be seated together?

10. Find in how many ways 9 persons may be seated at a round table

(1) supposing the seats distinguished;

(2) considering relative position only;

(3) considering relative position only except that two orders differing only in *direction* (or *sense*) are counted as one.

11. In how many ways may 11 persons be seated relatively at a round table if a certain two persons are not to be placed together?

12. In how many ways may 5 ladies and 5 gentlemen be seated relatively at a round table if no two ladies are to be seated together?

3. **The Fundamental Theorem of Combinations.** The theorem will be illustrated by finding the number of combinations of 5 letters a, b, c, d, e, taken 3 at a time. Denote the number sought by N. Take any combination *abc;* then by interchange of these letters we obtain in all 3.2.1 distinct permutations, each of three letters. The same is true of each of the possible combinations. Now no two distinct combinations can give rise to the same permutation, while from all possible combinations will be formed all possible permutations. Hence the total number of permutations of 5 letters 3 at a time is

$$N \times 3.2.1.$$

But this number is already known to be 5.4.3.

$$\therefore\ N \times 3.2.1 = 5.4.3$$

$$\therefore\ N = \frac{5.4.3}{3.2.1} = 10.$$

The student is recommended to write down these 10 combinations and *see* their relation to the 60 permutations of 5 letters 3 at a time.

The general position is: *To find the number of combinations of n things r at a time.*

Denote this number by ${}_nC_r$. Take any combination of the r things; then, by interchange of the r things of which it consists, it will give rise to $r\,!$ distinct permutations. The same is true of each combination. Further, no two combinations can give rise to the same permutation, while from all possible combinations will be formed all possible permutations. Therefore, the total number of permutations of n things r at a time is

$$ {}_nC_r \times r\,! $$

But this number is already known to be

$$ n(n-1)\ldots\ldots(n-r+1), \text{ or } \frac{n\,!}{(n-r)\,!}. $$

$$ \therefore\ {}_nC_r \times r\,! = n(n-1)\ldots\ldots(n-r+1) = \frac{n\,!}{(n-r)\,!}. $$

$$ \therefore\ {}_nC_r = \frac{n(n-1)\ldots\ldots(n-r+1)}{r\,!} = \frac{n\,!}{r\,!\,(n-r)\,!}. $$

Cor. The number of combinations of n things r at a time is equal to the number of combinations of n things $n-r$ at a time.

EXERCISES

1. From a company of 35 soldiers a picket of 5 soldiers has to be chosen; in how many ways is this possible?

2. Write down in full, *i.e.*, not employing the factorial symbol, the numbers denoted by ${}_7C_3$, ${}_9C_5$, ${}_5C_5$, and compute the value in each case.

3. From a case containing 15 books a person is to select 3 books; in how many ways may the selection be made?

4. If ${}_nC_{11} = {}_nC_7$ find n.

5. In a plane are 7 points; how many triangles may be formed with 3 of these points as angular points?

6. How many triangles can be formed having 3 of n given points in a plane as angular points?

7. From a committee of 15 ladies and 20 gentlemen, a sub-committee of 3 ladies and 4 gentlemen is to be chosen; in how many different ways may this be done?

8. How many diagonals has a heptagon? a quindecagon?

9. From 8 teachers and 60 pupils a committee of 3 teachers and 7 pupils is to be chosen; in how many ways may this be done?

In how many ways may the committee be chosen if a certain teacher and a certain 2 pupils are to serve on it?

10. Show that

$$_nC_r = \frac{n-r+1}{r} \cdot {_nC_{r-1}}$$

11. In how many ways may 15 different things be divided among 3 persons, each getting 5 things?

12. In how many ways may 15 different things be divided into 3 parcels of 5 each?

13. In how many ways may 10 different things be divided among A, B, C, D, if A and B are each to receive 2 things and C and D each 3 things?

14. In how many ways may 10 different things be divided into two parcels of 2 each and two parcels of 3 each?

15. Shew that

$$_nC_r + {_nC_{r-1}} = {_{n+1}C_r}.$$

16. In a plane are n points, p of which are in a straight line; how many triangles can be formed with 3 of the points as angular points?

17. Shew that

$$\frac{(2n)!}{n!} = 2^n(1.3.5.\dots.\overline{2n-1}).$$

18. In a plane are n points and through every two of them a straight line (produced indefinitely) is drawn. Find the number of points of intersection of these lines, exclusive of the n points.

19. How many combinations of 3 letters may be made from the letters of the word *Canada?*

20. How many combinations of 4 letters may be made from the letters of the word *Manhattan?*

21. Shew that mn different things may be divided among m persons, each receiving n things, in

$$\frac{(mn)!}{(n!)^m}$$

ways and that the number of ways in which the mn things may be divided into m parcels of n each is

$$\frac{(mn)!}{(n!)^n m!}$$

22. How many words of 3 consonants and 2 vowels may be formed from the letters of the word *amplitude?*

In how many of these will the vowels be separated?

23. Shew that the product of any r consecutive integers is divisible by r.

4. **Additional Theorems.** The theorems of the two preceding articles have been spoken of as fundamental. As the theorem on combinations has been derived from that on permutations, we might say that there is one essential proposition. We shall now derive certain further theorems either from those already given or by the methods already employed.

Theorem I. *The number of permutations of n things taken all at a time, p being alike and the rest unlike,* is

$$\frac{n!}{p!}.$$

Let N denote the number sought. Take any one of the permutations and in it suppose the p like things replaced by p things different from one another and from the rest. Then by permitting these p things without disturbing the rest we should form $p!$ distinct permutations. The same is true of each of the N permutations in question so that these would give rise to $N \times p!$ permutations. But these would be the $n!$ permutations among themselves of n unlike things.

$$\therefore \quad N \times p! = n!$$

$$\therefore \qquad N = \frac{n!}{p!}$$

Cor. The number of permutations of n things taken all at a time, p being of one kind, q of another kind, r of another kind, is

$$\frac{n!}{p!\,q!\,r!\ldots\ldots}$$

Theorem II. *The number of permutations of n things r at a time, when repetitions of each thing are allowed up to r times,* is n^r.

As in the earlier theorem suppose that there are r places to be filled each by one thing. The first may be filled in n ways; after it has been filled in any way the second may be filled in n ways, since the thing already placed may be repeated. Thus the number of ways of filling 2 places is n^2 and by continuing the reasoning we find that the number of ways of filling r places is n^r.

Theorem III. *The total number of ways in which a selection of one or more things from n things may be made is* $2^n - 1$.

The first thing may be taken or left, *i.e.*, it may be treated in 2 ways. In either case the second thing may be treated in 2 ways, so that the first two things may be treated in 2^2 ways. Similarly for the successive things and the number of ways of treating all the things is 2^n. As this includes the case in which all the things are left, and therefore no selection made, the result sought is $2^n - 1$.

Theorem IV. *The value of r for which the number of combinations of n things r at a time is greatest is* $\frac{n}{2}$ *if n is even and* $\frac{n-1}{2}$ *or* $\frac{n+1}{2}$ *when n is odd.*

It is readily seen that

$$ {}_nC_r = {}_nC_{r-1} \times \frac{n-r+1}{r}. $$

Therefore,

$$ {}_nC_r >=< {}_nC_{r-1} $$

$$ \text{as } \frac{n-r+1}{r} >=< 1 \; ; $$

$$ \therefore \text{ as } n-r+1 >=< r \; ; $$

$$ \therefore \text{ as } n+1 >=< 2r \; ; $$

$$ \therefore \text{ as } 2r <=> n+1 \; ; $$

$$ \therefore \text{ as } r <=> \frac{n+1}{2}. $$

(*a*) Let n be even. Then the greatest value the integer r can have to be less than $\frac{n+1}{2}$ is $\frac{n}{2}$; this, therefore, is the greatest value r can have if ${}_nC_r > {}_nC_{r-1}$. Thus, ${}_nC_r$ is greatest when $r = \frac{n}{2}$.

(*b*) Let n be odd. Then the greatest value r can have to be less than $\frac{n+1}{2}$ is $\frac{n-1}{2}$; this, therefore, is the greatest value r can have if ${}_nC_r > {}_nC_{r-1}$. Thus ${}_nC_r$ is greatest when $r = \frac{n-1}{2}$. But we must note, also, that if $r = \frac{n+1}{2}$, *i.e.*, greater by 1 than $\frac{n-1}{2}$, then will ${}_nC_r = {}_nC_{r-1}$ so that the number of combinations $\frac{n-1}{2}$ at a time is the same as the number $\frac{n+1}{2}$ at a time and each is greater than the number given by any other value of r.

EXAMPLES

1. How many numbers of 5 figures each may be made from the nine digits

   (1) if digits may not be repeated;

   (2) if digits may be repeated?

   In (1) how many numbers begin with 5 and end with 7, and how many have 5 as their middle digit?

   In (2) how many numbers begin and end with 5, and how many have 5 as their middle digit?

2. Find the total number of possible combinations of $p+q$ things of which p are of one kind and q of another.

3. Find the number of combinations of 4 letters and the number of permutations of 4 letters that can be formed from the letters of the word *terrestrial.*

4. Shew that the number of combinations, n at a time, of $2n$ things of which n are alike exceeds by 1 the total number of combinations of n things.

5. In a city are m streets running north and south and n streets running east and west. Find in how many different ways the direct journey from the north-east corner of the city to the south-west corner may be made.

6. If twenty persons are seated at a round table, in how many ways may three of them be selected if no two of these three are to be seated together?

7. In how many ways can p positive signs and n negative signs be arranged in a straight line if no two negative signs are to be together?

8. Find in how many ways 8 like things may be given to 5 persons

(1) if there is made no restriction as to the mode of distribution;

(2) if each person is to receive at least one thing;

(3) if each person is to receive one thing and no person more than two.

CHAPTER VIII

THE BINOMIAL THEOREM

1. **Preliminary.** By actual multiplication we know the expansion of $(a+x)^2$, $(a+x)^3$, $(a+x)^4$, and it may be a few higher powers of the *binomial* $a+x$. Thus, for example,

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3.$$

The question arises whether it is possible to obtain a rule for writing down the expansion of a binomial to *any positive integral* power and whether we can speak of an expansion and a rule for an expansion if the exponent is a *positive fraction* or a *negative integer or fraction.*

It will be seen that a rule is furnished by the **Binomial Theorem.**

The case first to be examined is that in which the exponent is a positive integer. It will be well, before treating the general problem, to obtain the expansion of say $(a+x)^3$ by a method other than *formal* multiplication. Plainly $(a+x)^3$ is the product of the three factors,

$$a+x,\ a+x,\ a+x.$$

Each term of the product will be of three dimensions, one dimension or letter being furnished by each factor, so that all possible terms are

$$a^3,\ a^2x, ax^2,\ x^3$$

except that coefficients are wanting. The term a^3 can and will be formed in only one way, namely by taking a from each factor; its coefficient is therefore 1. The term a^2x will be formed by taking x from one factor and with it the a from each of the other two factors; x can be chosen from one of three factors in 3 ways so that the term a^2x can and will be formed in 3 ways and its coefficient is therefore 3. The term ax^2 will be formed by taking x from two factors, and with these two x's the a from the remaining factor; x can be chosen from two of three factors in $\frac{3.2}{1.2}$ or 3 ways and 3 is the coefficient of ax^2.

The coefficient of x^3 is seen to be 1. Thus we have

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3.$$

EXERCISES

1. Find in the manner just given the expansions of

$$(a+x)^4,\ (a+x)^5,\ (a+x)^6.$$

2. Find in like manner the expansions of

$$(a-x)^3,\ (a+2b)^5,\ (2a-3b)^7.$$

3. How many terms in the expansions of

$$(a+x)^9,\ (a-x)^{15},\ (2a+3b)^{17}?$$

2. **The Binomial Theorem for a Positive Integral Exponent.** It is proposed to find the expansion of $(a+x)^n$ where n is a positive integer.

By $(a+x)^n$ is meant the product of n factors

$$a+x,\ a+x,\ \ldots\ldots\ a+x.$$

Each term will be of n dimensions, one dimension or letter coming from each factor, so that all possible terms are

$$a^n,\ a^{n-1}x^1,\ \ldots\ldots\ a^{n-r}x^r,\ \ldots\ldots\ x^n,$$

except that the coefficients remain to be found.

The term a^n can and will be formed in only one way, namely, by taking a from each factor; hence its coefficient is 1.

The term $a^{n-1}x$ will be formed by taking x from any one factor with the a from each of the remaining factors; as x can be chosen in n ways, the coefficient of $a^{n-1}x$ is n.

The *general* term $a^{n-r}x^r$, the $(r+1)$th in order, will be formed by taking x from any r of the factors with the a from each of the remaining factors; as the r factors which are to furnish x can be chosen in ${}_nC_r$ ways, the coefficient of $a^{n-r}x^r$ is

$${}_nC_r, \text{ or } \frac{n(n-1)\ldots\ldots(n-r+1)}{1.2\ldots\ldots r}, \text{ or } \frac{n!}{(n-r)!\,r!}.$$

By giving to r the values $1, 2, \ldots\ldots n$, we obtain the coefficients of all the terms after the first. Hence we have

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 + \ldots\ldots$$
$$+ \frac{n(n-1)\ldots\ldots(n-r+1)}{1.2\ldots\ldots r}a^{n-r}x^r + \ldots\ldots + x^n,$$

and the required rule for the expansion has been found.

Cor. 1. The sum of the coefficients in the expansion of $(a+x)^n$ *is* 2^n. (Found by putting $a=1$, $x=1$.)

Cor. 2. The sum of the odd coefficients is equal to the sum of the even coefficients. (Found by putting $a=1$, $x=-1$.)

Cor. 3. The coefficients of terms equidistant from the beginning and the end are the same.

NOTE :—

(1) The number of terms is $n+1$, so that if n is even there is a middle term, and if n is odd there are two middle terms.

(2) For convenience the expansion is frequently written thus:

$$(a+x)^n = a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \ldots\ldots + \binom{n}{r}a^{n-r}x^r + \ldots\ldots + x^n,$$

where $\binom{n}{r}$ denotes $\frac{n(n-1)\ldots\ldots(n-r+1)}{1.2\ldots\ldots r}$. We may agree to denote the first and the last coefficient, namely 1, by $\binom{n}{0}$.

(3) $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots\ldots + \binom{n}{r}x^r + \ldots\ldots + x^n.$

(4) $(a-x)^n = a^n - \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 - \ldots\ldots + (-1)^r\binom{n}{r}a^{n-r}x^r + \ldots\ldots + (-1)^n x^n.$

Ex. 1. Find the middle term of $(2x-3y)^{10}$.

There are in all 11 terms so that the middle term is the 6th, *i.e.*, the term involving y^5. It is therefore

$$\frac{10.9.8.7.6}{1.2.3.4.5}(2x)^5(-3y)^5, \text{ or } -252.32.243x^5y^5.$$

Ex. 2. Find the sum of the squares of the coefficients in the expansion of $(1+x)^n$.

Denote the coefficient of x^r by c_r; then $c_0=1=c_n$ and $c_r=c_{n-r}$.

$$\therefore\ (1+x)^n = c_0 + c_1\ x + c_2\ x^2 + \ldots\ldots + c_{n-1}x^{n-1} + c_n x^n.$$

$$\therefore\ (1+x)^n = c_n + c_{n-1}x + c_{n-2}x^2 + \ldots\ldots + c_1\ x^{n-1} + c_0 x^n.$$

In the product of the series the coefficient of x^n is

$$c_0{}^2 + c_1{}^2 + c_2{}^2 + \ldots\ldots + c_n{}^2$$

and this then is equal to the coefficient of x^n in $(1+x)^{2n}$ and therefore to $\dfrac{(2n)!}{n!\,n!}$.

The student is recommended to work the example by finding the coefficient of x^0 in the product of the expansions of $(1+x)^n$ and $\left(1+\dfrac{1}{x}\right)^n$.

Ex. 3. If

$$(1+x)^m = a_0 + a_1 x + \ldots\ldots + a_r x^r + \ldots\ldots + a_m x^m,$$
$$(1+x)^n = b_0 + b_1 x + \ldots\ldots + b_r x^r + \ldots\ldots + b_n x^n,$$

find the value of

$$b_0 a_r + b_1 a_{r-1} + b_2 a_{r-2} + \ldots\ldots + b_r a_0.$$

This is evidently the coefficient of x^r in the product of the two series, and, therefore, to the coefficient of x^r in $(1+x)^{m+n}$ which is equal to

$$\frac{(m+n)(m+n-1)\ldots\ldots(m+n-r+1)}{1.2\ldots\ldots r} \text{ or } \binom{m+n}{r}.$$

$$\therefore \binom{m}{r}\binom{n}{0}+\binom{m}{r-1}\binom{n}{1}+\binom{m}{r-2}\binom{n}{2}+\ldots\ldots+\binom{m}{0}\binom{n}{r}=\binom{m+n}{r}.$$

EXERCISES

1. Write out the complete expansions of:

$$(a+b)^8,\ (a-b)^7,\ (a+2b)^5,\ (1-\tfrac{1}{2}x)^6,\ (1+\tfrac{2}{3}x)^4.$$

2. Find the middle term of the expansions of:

$$(x+y)^6,\ (x-y)^{20},\ (2a-3y)^{14},\ (\tfrac{1}{2}x-\tfrac{1}{3}y)^{12}.$$

3. Find the two middle terms of:

$$(2a-3b)^5,\ (x-y)^{23},\ (x^2-y^2)^{17}.$$

4. Write down the general term, *i.e*, the $(r+1)$th term, of:

$$(1-x)^n,\ (1+3x)^m,\ (a-2x)^n.$$

3. The Binomial Theorem for Fractional or Negative Exponents. The question now arises whether an expansion can be found for $(a+x)^n$ when n is fractional or negative, and, if so, whether the rule established for the case in which the exponent is a positive integer is applicable. We shall first examine certain examples.

Ex. 1. Examine whether an expansion in ascending powers of x can be found for $(1-x)^{-1}$.

We have $(1-x)^{-1}=\dfrac{1}{1-x}=1\div(1-x)$.

Let the division be performed:

$$\begin{array}{llll}
1-x)1 & & & (1+x+x^2+\dots \\
\quad \underline{1-x} & & & \\
\qquad +x & & & \\
\qquad \underline{+x-x^2} & & & \\
\qquad\qquad +x^2 & & & \\
\qquad\qquad \underline{+x^2-x^3} & & & \\
\qquad\qquad\qquad +x^3 & & &
\end{array}$$

Plainly the operation of division will not terminate. At any step we may write down a value of $(1+x)^{-1}$, *in part* a series. Thus we may say

$$(1-x)^{-1}=1+x+x^2+\frac{x^3}{1-x}.$$

We are led to say

$$(1-x)^{-1}=1+x+x^2+x^3+\dots\dots in\ inf.$$

And we have indeed seen that the limit of this series is $(1-x)^{-1}$, *if x is numerically less than unity.*

Let us now apply the binomial rule, *shewn* to hold for a positive integral exponent. This would give

$$1+(-1)(-x)+\frac{(-1)(-2)}{1.2}(-x)^2+\dots+\frac{(-1)(-2)\dots(-1-r+1)}{1.2.3\dots r}(-x)^r+\dots$$

or when reduced

$$1+x+x^2+x^3+\dots\dots$$

a series which will not terminate. The result is in agreement with the expansion found by division.

Ex. 2. Examine whether an expansion in ascending powers of x can be found for $(1-x)^{-2}$.

We have

$$(1-x)^{-2}=\frac{1}{(1-x)^2}=\frac{1}{1-2x+x^2}.$$

Let the division be performed :

$$\begin{array}{l}
1-2x+x^2)\ 1 \qquad\qquad (1+2x+3x^2+ \\
\qquad\qquad\ \ \underline{1-2x+x^2} \\
\qquad\qquad\qquad 2x-x^2 \\
\qquad\qquad\qquad \underline{2x-4x^2+2x^3} \\
\qquad\qquad\qquad\qquad 3x^2-2x^3 \\
\qquad\qquad\qquad\qquad \underline{3x^2-6x^3+3x^4} \\
\qquad\qquad\qquad\qquad\qquad 4x^3-3x^4
\end{array}$$

It is seen that the division will not terminate and it may also be seen that the law of terms in the quotient will continue. This last, however, is more easily seen if we note that

$$(1-x)^{-2}=(1-x)^{-1}\div(1-x),$$

and assume that it is permissible to divide an infinite series. We then have

$$(1-x)^{-2}=(1+x+x^2+\ldots\ldots \textit{ in inf.})\div(1-x).$$

Let the division be undertaken :

$$\begin{array}{l}
1-x)\ 1+x+x^2+x^3+\ldots\ldots\ (1+2x+3x^2+\ldots. \\
\qquad\ \ \underline{1-x} \\
\qquad\qquad 2x+\ x^2 \\
\qquad\qquad \underline{2x-2x^2} \\
\qquad\qquad\qquad 3x^2+x^3 \\
\qquad\qquad\qquad\quad \ldots\ldots
\end{array}$$

The law of terms in the quotient is now evident.

Now, let us apply the binomial rule to $(1-x)^{-2}$. This would give

$$1+(-2)(-x)+\frac{(-2)(-3)}{1.2}(-x)^2+\ldots+\frac{(-2)(-3)\ldots(-3-r+1)}{1.2\ \ldots\ r}(-x)^r+\ldots$$

which reduces to

$$1+2x+3x^2+\ldots\ldots+(r+1)\,x^r+\ldots\ldots$$

The result is in agreement with what was previously seen.

The application of the rule here as in the earlier example furnishes an infinite series.

Referring to Ex. 4, p. 85, we see that we can find the sum of n terms of the series

$$1+2x+3x^2+\ldots..$$

and *if x is numerically less than unity*, it can be shewn that the limit of the infinite series is $(1-x)^{-2}$.

Ex. 3. Find an expression in ascending powers of x for $(1+x)^{\frac{1}{2}}$.

We have $(1+x)^{\frac{1}{2}}=\sqrt{(1+x)}$. Let the square root be extracted,

$$\begin{array}{ll}
 & 1+x\left(1+\frac{x}{2}-\frac{x^2}{8}\right. \\
 & \underline{1\quad} \\
2+\frac{x}{2} & \quad +x \\
 & \underline{\quad +x+\frac{x^2}{4}} \\
2+x-\frac{x^2}{8} & \qquad -\frac{x^2}{4} \\
 & \qquad \underline{-\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64}} \\
 & \qquad\qquad\qquad \ldots..
\end{array}$$

If the binomial rule were applied we should obtain

$$1+\tfrac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}x^2+\ldots..$$

or

$$1+\frac{x}{2}-\frac{x^2}{8}+\ldots..$$

a result which agrees with the value found for $(1+x)^{\frac{1}{2}}$. It is plain that the series will not terminate, and it could be shewn that the infinite series has a meaning only *when x is numerically less than unity.*

It is also to be noted that $(1+x)^{\frac{1}{2}}$ has two values. In finding the square root we might have started with -1 as well as with $+1$ and the two values found would differ only in sign. The binomial rule gives only one of the roots.

A study of the preceding examples would seem to lead to the conclusion that the Binomial Theorem is valid, under certain restric-

tions, for all integral or fractional positive or negative exponents. The student who is not making a special study of mathematics may, for the case of a fractional or negative exponent, assume its truth as thus stated:

If

(1) *n is any positive integer and x any number,*

or if

(2) *n is a positive fraction or a negative integer or fraction while x is numerically less than unity,*

then

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 + \ldots\ldots + \frac{n(n-1)\ldots\ldots(n-r+1)}{1.2\ldots\ldots r}x^r + \ldots\ldots$$

It is to be noted that if n is a positive integer the series terminates through the appearance of a factor zero in the numerator of a term; if n is not a positive integer no such factor can appear and the series will not terminate.

If it is a question of the expansion of $(a+x)^n$ where n is fractional or negative, it is well, as a rule, to regard this as $a^n\left(1+\frac{x}{a}\right)^n$.

In Art. 5 of this chapter will be given a proof of this theorem which may, if desired, be now studied. The following examples, however, do not depend upon the *method of proof* but only on the *result* which as said may be assumed.

Ex. 1. Expand $(1-x)^{-3}$.

It follows from the general theorem that

$$(1-x)^{-3} = 1 + (-3)(-x) + \frac{(-3)(-4)}{1.2}(-x)^2 + \ldots\ldots$$

The general term is

$$\frac{(-3)(-4)\ldots\ldots(-3-r+1)}{1.2\ldots\ldots r}(-x)^r.$$

This equals

$$(-1)^r.\frac{3.4\ldots\ldots(r+2)}{1.2\ldots\ldots r}(-x)^r, \text{ or } \frac{(r+1)(r+2)}{1.2}x^r.$$

$$\therefore\ (1-x)^{-3} = 1 + 3x + \frac{3.4}{1.2}x^2 + \ldots\ldots + \frac{(r+1)(r+2)}{1.2}x^r + \ldots.$$

Ex. 2. Expand $(a+x)^{-2}$

$$(a+x)^{-2}=a^{-2}.\left(1+\frac{x}{a}\right)^{-2}=\frac{1}{a^2}\cdot\left\{1+(-2)\frac{x}{a}+\frac{(-2)(-3)}{1.2}\cdot\left(\frac{x}{a}\right)^2+\ldots\ldots\right\}$$

The general term of the expansion within the brackets is

$$\frac{(-2)(-3)\ldots\ldots(-2-r+1)}{1.2\ldots\ldots r}\cdot\left(\frac{x}{a}\right)^r \text{ or } (-1)^r.\ (r+1)\frac{x^r}{a^r}$$

$$\therefore\ (a+x)^{-2}=\frac{1}{a^2}-2\frac{x}{a^3}+3\frac{x^2}{a^4}-\ldots\ldots+(-1)^r(r+1).\frac{x^r}{a^{r+2}}+\ldots\ldots$$

Ex. 3. Find the coefficient of x^r in the expansion of $\left(\frac{1+x}{1-x}\right)^2$.

$$\left(\frac{1+x}{1-x}\right)^2=(1+x)^2(1-x)^{-2}=(1+2x+x^2)(1+2x+3x^2+\ldots\ldots+\overline{r+1}\,x^r+\ldots.)$$

In this last product the coefficient of x^r is seen to be

$$(r+1)+2.r+(r-1) \text{ or } 4r.$$

This result may be obtained otherwise in noting that

$$\left(\frac{1+x}{1-x}\right)^2=\frac{(1-x)^2+4x}{(1-x)^2}=1+4x(1-x)^{-2}.$$

Ex. 4. Find the sum of the first $r+1$ coefficients in the expansion of $(1-x)^n$ where n may be any positive or negative integer or fraction.

Let

$$(1-x)^n = c_0+c_1x+c_2x^2+\ldots\ldots+c_rx^r+\ldots\ldots$$

Also we have

$$(1-x)^{-1}=1+x+x^2+\ldots\ldots+x^r+\ldots\ldots$$

Then $c_0+c_1+\ldots\ldots+c_r$ equals the coefficient of x^r in the expansion of the product $(1-x)^n(1-x)^{-1}$ or $(1-x)^{n-1}$. This coefficient equals

$$\frac{(n-1)(n-2)\ldots\ldots(n-1-r+1)}{1.2\ldots\ldots\ldots r} \text{ or } \frac{(n-1)(n-2)\ldots\ldots(n-r)}{r\,!}.$$

EXERCISES

1. The following special expansions are so important that the student should be familiar with them. They are put as exercises and the general term in each case is to be found.

(1) $(1-x)^{-1} = 1+x+x^2+x^3+x^4+\ldots\ldots$

(2) $(1+x)^{-1} = 1-x+x^2-x^3+x^4-\ldots\ldots$

(3) $(1-x)^{-2} = 1+2x+3x^2+4x^3+5x^4+\ldots\ldots$

(4) $(1-x)^{-3} = 1+3x+\frac{3.4}{1.2}x^2+\frac{4.5}{1.2}x^3+\frac{5.6}{1.2}x^4+\ldots\ldots$

(5) $(1-x)^{-4} = 1+4x+\frac{4.5}{1.2}x^2+\frac{4.5.6}{1.2.3}x^3+\frac{5.6.7}{1.2.3}x^4+\ldots\ldots$

(6) $(1-x)^{-m} = 1+mx+\frac{m(m+1)}{1.2}x^2+\frac{m(m+1)(m+2)}{1.2.3}x^3+\ldots\ldots$

2. Find the general term in the expansion of each of the following :

$$(1-x)^{\frac{1}{2}},\ (1-x)^{-\frac{1}{2}},\ (a+x)^{\frac{3}{2}},\ (2a-3x)^{-5}.$$

3. Find the first negative term in the expansion of $(1+x)^{\frac{9}{4}}$.

4. Expand to 5 terms in a series of powers of the fractions appearing in the binomial :

$$\left(1-\frac{1}{10}\right)^{\frac{1}{2}},\ \left(1+\frac{1}{3}\right)^{-3},\ \left(1+\frac{1}{2}\right)^{-\frac{1}{3}}.$$

5. Find the general term in the expansion of each of the following :

$$(1+x)^{\frac{1}{3}},\ (1+3x)^{-\frac{2}{3}},\ (1-x)^{-\frac{1}{5}}.$$

6. Find the coefficient of x^r in the expansion of each of the following :

$$\frac{1+x}{(1-x)^3},\ \frac{(1+x)^2}{(1-x)^3},\ \frac{(1+x)^3}{(1-x)^2}.$$

7. Expand to four terms each of the following :

$$(1-x)^{\frac{1}{2}}\ (1-x)^{\frac{1}{3}},\ (1+x)^{\frac{1}{3}}\ (1+x)^{\frac{1}{2}}\ (1+2x)^{\frac{2}{3}}.$$

8. Find the sum of the first $r+1$ coefficients of the expansion of $\frac{1-x}{(1+x)^m}$.

9. Prove that the coefficient of x^r in the expansion of $(1-4x)^{-\frac{1}{2}}$ is $\frac{(2r)!}{(r!)^2}$

10. Recover the power of a binomial which leads to each of the following:

(1) $1+2\left(\frac{1}{3}\right)+3\left(\frac{1}{3}\right)^2+4\left(\frac{1}{3}\right)^3+\ldots\ldots$

(2) $1-\frac{1}{2}\cdot\frac{1}{2}+\frac{1.3}{2.4}\cdot\frac{1}{2^2}-\frac{1.3.5}{2.4.6}\cdot\frac{1}{2^3}+\ldots\ldots$

(3) $1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\ldots\ldots$

(4) $1+\frac{1}{5}+\frac{1.4}{1.2}\cdot\frac{1}{5^2}+\frac{1.4.7}{1.2.3}\cdot\frac{1}{5^3}+\ldots\ldots$

4. **Approximations.** When in a problem the result is given in the form of an infinite series it is in general necessary, for practical purposes, to accept the **approximation** furnished by some definite and small number of terms. The following examples will illustrate this fact. The question of the degree of the approximation will not be here investigated, the student being referred to the excellent Algebra of Chrystal, Vol. II, p. 192, *et seq.*

Ex. 1. A cube of copper of edge 1 in. at 0°C. is brought to a temperature of 1°C. It is found that each edge has been increased in length 0·000017 in.; find the increase in volume. The volume at 1°C. is $(1+0{\cdot}000017)^3$ cubic in. and this equals

$$(1+3\times0{\cdot}000017+3\times0{\cdot}000017^2+0{\cdot}000017^3) \text{ c. in.}$$

The last two terms are very small compared with the second term and the volume to a degree of approximation that makes it *practically correct* is $(1+3\times0{\cdot}000017)$ c. in., so that the expansion in volume is $3\times0{\cdot}000017$ or 0·000051 c. in.

Here 0·000017 is the coefficient of linear expansion of copper and 0·000051 the coefficient of cubical expansion.

In like manner if the coefficient of linear expansion of a given metal is x the coefficient of cubical expansion is $3x$.

Ex. 2. Find approximately the square root of 99.

$$\sqrt{99}=(100-1)^{\frac{1}{2}}=100^{\frac{1}{2}}\left(1-\frac{1}{100}\right)^{\frac{1}{2}}=10\left(1-\frac{1}{100}\right)^{\frac{1}{2}}$$

$$=10\left(1-\frac{1}{2}\cdot\frac{1}{100}-\frac{1}{8}\cdot\frac{1}{100^2}-\frac{1}{16}\cdot\frac{1}{100^3}-\ldots\ldots\right)$$

The terms within the brackets become small very rapidly, and if we take the first three terms as an approximation, the result is 9·949875, which is correct

to 5 places of decimals, the root having the digit 4 in the sixth place. By taking four terms we get a result correct to eight places of decimals.

Ex. 3. If x is a quantity so small that the cubes and higher powers of x may be neglected, find the approximation to the value of

$$\frac{(1+2x)^{\frac{1}{2}}(1-x)^{\frac{3}{2}}}{(2+3x)^2}.$$

$$\text{This fraction} = \frac{(1+2x)^{\frac{1}{2}}(1-x)^{\frac{3}{2}}\left(1+\frac{3x}{2}\right)^{-2}}{4}$$

$$= \frac{\left(1+x-\frac{x^2}{2}\right)\left(1-\frac{3x}{2}+\frac{3x^2}{8}\right)\left(1-3x-\frac{27x^2}{4}\right)}{4}, \text{ approximately}$$

$$= \frac{\left(1-\frac{1}{2}x+\frac{13}{8}x^2\right)\left(1-3x-\frac{27x^2}{4}\right)}{4}, \quad \text{"}$$

$$= \frac{1-\frac{7x}{2}-\frac{29x^2}{8}}{4}, \quad \text{"}$$

$$= \frac{1}{4}-\frac{7x}{8}-\frac{29x^2}{32}, \text{ neglecting powers higher than the third.}$$

EXERCISES

1. Find approximately the square root of each of the following numbers :

$$24, \; 80, \; 620, \; 224, \; 1220.$$

2. Find approximately the value of each of the following :

$$63^{\frac{1}{3}}, \; 97^{\frac{1}{2}}, \; 1 \div 47^{\frac{1}{2}}, \; 240^{\frac{1}{5}} \; 1 \div 127^{\frac{1}{3}}.$$

3. If the coefficient of cubical expansion of a certain metal is 0 000078, find the coefficient of linear expansion.

4. If x is so small that its second and higher powers may be neglected find the value of each of the following :

$$(1+2x)^{\frac{1}{2}}(1-3x)^{\frac{2}{3}}, \; (4+x)^{-\frac{1}{2}}(1-x)^{-3}, \; (8+x)^{-\frac{1}{3}}(1+x)^{\frac{2}{7}},$$

$$\frac{(1+2x)^{\frac{1}{2}}+(1-3x)^{-\frac{2}{3}}}{(1+4x)^{\frac{3}{4}}+(1+5x)^{\frac{1}{5}}}, \; \frac{(5+9x)^{-1}+(3+2x)^{-2}}{(1+x)^{\frac{1}{3}}}.$$

5. **Proof of the Binomial Theorem** *for a Positive Fractional Exponent and for a Negative Integral or Fractional Exponent.*

The following proof of the theorem, under stated assumptions, for the case in which the exponent is fractional or negative, is now added.

The series

$$1+\frac{m}{1}x+\frac{m(m-1)}{1.2}x^2+\ldots\ldots+\frac{m(m-1)\ldots\ldots(m-r+1)}{1.2\ldots\ldots r}x^r+\ldots\ldots$$

will terminate if m is a positive integer and its value is then $(1+x)^m$. If m is not a positive integer no factor 0 can be introduced into the numerator of any term so that the series will not terminate; this infinite series has a finite limit when $|x|<1$, a fact which will be *assumed*, and under this supposition as to the values of m and x, it is proposed to find the value of the series. The value of the series depends on that of m; it is then a certain function of m which will be denoted by $f(m)$. Thus if m is a positive integer $f(m)=(1+x)^m$. Then if $\binom{m}{r}$ is written for $\frac{m(m-1)\ldots\ldots(m-r+1)}{1.2\ldots\ldots r}$ whatever be the value of m,

$$f(m)=1+\binom{m}{1}x+\binom{m}{2}x^2+\ldots\ldots+\binom{m}{r}x^r+\ldots\ldots$$

Also, if m is replaced by n,

$$f(n)=1+\binom{n}{1}x+\binom{n}{2}x^2+\ldots\ldots+\binom{n}{r}x^r+\ldots\ldots$$

Assume now that it is permissible to multiply two infinite series with finite limit as if they were polynomials; then

$$f(m)\,.\,f(n)=1+\left\{\binom{m}{1}+\binom{n}{1}\right\}x+\text{an infinite series of higher powers of }x.$$

In this product, the coefficient of x^r is

$$\binom{n}{r}+\binom{m}{1}\binom{n}{r-1}+\ldots\ldots\binom{m}{r}\binom{n}{r}.$$

This series of fractions involving m and n may be added to form one fraction and the way in which m and n appear in the result will be the

same whatever values m and n have. But when m and n are positive integers (See *Ex. 3*, p. 102), the sum is

$$\binom{m+n}{r}, \text{ or } \frac{(m+n)(m+n-1)\ldots\ldots(m+n-r+1)}{1.2\ldots\ldots r}.$$

which must then be the sum whatever be m and n. Therefore, whatever be the values of m and n,

$$f(m).f(n) = 1+(m+n)x+\frac{(m+n)(m+n-1)}{1.2}x^2+\ldots\ldots$$

$$= 1+\binom{m+n}{1}x+\binom{m+n}{2}x^2+\ldots\ldots+\binom{m+n}{r}x^r+\ldots\ldots$$

$\therefore\ f(m).f(n)=f(m+n).$

Hence, also,

$$f(m).f(n).f(p)=f(m).f(n+p)=f(m+n+p),$$

and generally,

$$f(m).f(n)\ldots\ldots f(t)=f(m+n+\ldots\ldots+t).$$

Suppose that here there are q factors and put $m, n, \ldots\ldots, t$ each equal to $\frac{p}{q}$ where p and q are positive integers. Then

$$\left\{f\left(\frac{p}{q}\right)\right\}^r=f\left(\frac{p}{q}\times q\right)=f(p)$$

$$=(1+x)^p, \text{ since } p \text{ is a positive integer.}$$

$$\therefore\ f\left(\frac{p}{q}\right)=(1+x)^{\frac{p}{q}}.$$

Hence, if $f\left(\frac{p}{q}\right)$ is written at length,

$$(1+x)^{\frac{p}{q}}=1+\frac{p}{q}x+\frac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{1.2}x^2+\ldots+\frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\ldots\left(\frac{p}{q}-r+1\right)}{1.2\ldots\ldots\ldots r}x^r+\ldots$$

and the binomial rule applies to the case in which the exponent is a positive fraction.

As there are more than one qth root (q indeed) of which one or at most two are real, it is necessary to determine which qth root is given by the expansion. Let x grow to the value 0; then $(1+x)^{\frac{p}{q}}$ becomes $1^{\frac{p}{q}}$ and the series becomes 1; hence, for these to be equal, the qth root taken must be the arithmetical root.

Next let m be a negative integer or fraction, and put it equal to $-m'$ where m' is positive. Then

$$f(-m').f(m')=f(-m'+m')=f(0)=1$$

$$\therefore \quad f(-m')=\frac{1}{f(m')}.$$

But m' being a positive integer or fraction $f(m')=(1+x)^{m'}$

$$\therefore \quad f(-m')=\frac{1}{(1+x)^{m'}}=(1+x)^{-m'}$$

$$\therefore \quad (1+x)^m=f(m),$$

or if $f(m)$ is written at length

$$(1+x)^m=1+\frac{m}{1}x+\frac{m(m-1)}{1.2}x^2+\ldots\ldots+\frac{m(m-1)\ldots\ldots(m-r+1)}{1.2\ldots\ldots r}x^r+\ldots\ldots$$

Hence the binomial rule applies to the case in which m is a negative integer or fraction.

6. **The number of Homogeneous Products.** Let $a, b, c, \ldots\ldots, h$, be n given letters, and consider the product of the n infinite series:

$$1+ax+a^2x^2+\ldots\ldots+a^rx^r+\ldots\ldots$$
$$1+bx+b^2x^2+\ldots\ldots+b^rx^r+\ldots\ldots$$
$$1+cx+c^2x^2+\ldots\ldots+c^rx^r+\ldots\ldots$$
$$\ldots\ldots$$
$$1+hx+h^2x^2+\ldots\ldots+h^rx^r+\ldots\ldots$$

In this product the coefficient of x^r will be the sum of all possible terms of r dimensions that can be made from the n given letters and repetitions of them, *i.e.*, will be the sum of the homogeneous products of n things r at a time. If each of $a, b, c, \ldots\ldots, h$ be put equal to 1, each term in

this sum is 1, and the sum is the *number* of such products. But when the letters are thus replaced by unity, the product of infinite series becomes $(1+x+x^2+\ldots\ldots)^n$ or $(1-x)^{-n}$, and in this the coefficient of x^r is

$$\frac{n(n+1)\ldots\ldots(n+r-1)}{1.2.\ldots\ldots r},$$

and this is therefore the *number* of homogeneous products of n things r at a time.

7. **The Numerically Greatest Term.** It is proposed to find the number of the term or terms of greatest numerical value in the expansion of $(1+x)^n$. The $(r+1)$th term is formed from the rth by multiplying by

$$\frac{n-r+1}{r}\,.\,x \text{ or } \left(\frac{n+1}{r}-1\right)x$$

which may be called the *multiplier for the* $(r+1)$*th term.* As we are concerned only with the numerical value of terms, we need consider only the numerical value of this multiplier, and may then suppose x to be positive.

I. Suppose n a positive integer.

Then since r cannot be greater than $n+1$ the multiplier is always positive. Now

$$\text{the } (r+1)\text{th term} > = < \text{the } r\text{th term}$$

$$\text{as } \left(\frac{n+1}{r}-1\right)x > = < 1$$

$$\therefore \text{ as } \frac{n+1}{r} > = < \frac{1}{x}+1$$

$$\therefore \text{ "} \quad \frac{r}{n+1} < = > \frac{x}{x+1}$$

$$\therefore \text{ "} \quad r < = > \frac{(n+1)x}{x+1}.$$

As long as r is less than this value the terms continue to increase and, when r passes this value, to diminish. If $\dfrac{(n+1)x}{x+1}$ is an integer p, the multiplier for $r=p$ is 1, and the pth and $(p+1)$th terms are equal and

greater than any other term. If $\frac{(n+1)x}{x+1}$ is fractional and between the integers p and $p+1$, then $r=p$ is the last value that makes the multiplier for the $(r+1)$th term greater than unity and the $(p+1)$th term is the greatest.

II. Suppose n a positive fraction, or a negative integer or fraction. We have then to take x less than 1.

(1) Let n be a positive or negative proper fraction.

Then $$0<n+1<2$$

$$\therefore \quad 0<\frac{n+1}{r}<2, \text{ for all values of } r$$

$$\therefore \quad -1<\frac{n+1}{r}-1<1, \text{ for all values of } r$$

Thus, x being less than 1, $\left(\frac{n+1}{r}-1\right)x$ is numerically less than 1, so that the multiplier for the $(r+1)$th term is always numerically less than 1 and the first term is the greatest.

(2) Let $n=-1$.

The expansion is then $1+x+x^2+\ldots\ldots$, and the first term is the greatest.

(3) Let $n<-1$:

Put $n=-m$ where m is positive and greater than 1. The multiplier is then numerically $\left(\frac{m+r-1}{r}\right)x$ or $\left(\frac{m-1}{r}+1\right)x$. Then

$$\text{the } (r+1)\text{th term} >=< \text{the } r\text{th term}$$

$$\text{as } \left(\frac{m-1}{r}+1\right)x >=< 1$$

$$\therefore \quad '' \quad \frac{m-1}{r} >=< \frac{1}{x}-1$$

$$\therefore \quad '' \quad \frac{r}{m-1} <=> \frac{x}{1-x}$$

$$\therefore \quad '' \quad r <=> \frac{(m-1)x}{1-x}, \text{ or } \frac{-(n+1)x}{1-x}$$

Therefore as in I, if this last value is an integer p, the pth and $(p+1)$th terms are equal and greater than any other term, while, if it lies between the integers p and $p+1$, the $(p+1)$th term is the greatest term.

(4) Let $n > 1$.

When $r > n+1$ it is plain that the multiplier $\left(\frac{n+1}{r} - 1\right) x$ is numerically less than unity, so that the greatest term is among those for $r < n+1$, since n not being an integer, r cannot equal $n+1$. The multiplier is in this case positive. As before we find that

$$(r+1)\text{th term} >=< r\text{th term},$$

as
$$r >=< \frac{(n+1)x}{1+x},$$

and that, if this value is an integer p, the pth and $(p+1)$th terms are equal, and are the greatest terms, while if this value lies between the integers p and $p+1$, the $(p+1)$th term is the greatest.

EXERCISES

1. Find the number of the greatest term in each of the following expansions, reproducing the reasoning of this article :

(1) $(1-x)^{-3}$ for $x=\frac{3}{4}$.

(2) $(1+x)^{-\frac{2}{3}}$ for $x=\frac{5}{7}$.

(3) $(1+x)^{9}$ for $x=\frac{1}{2}$ and for $x=\frac{3}{2}$.

(4) $(1+x)^{\frac{13}{2}}$ for $x=\frac{2}{3}$.

In each case write down the multiplier for the 2nd, 3rd, ... terms until the greatest term is reached.

2. Find the number of the greatest term in the expansion of each of the following :

(1) $(1+x)^{-2}$ for $x=\frac{5}{6}$.

(2) $(1+x)^{13}$ for $x=\frac{5}{4}$.

(3) $(1+x)^{-7}$ for $x=\frac{4}{5}$.

(4) $(1+x)^{\frac{37}{3}}$ for $x=\frac{8}{9}$.

EXAMPLES

(NOTE.—When, in the following examples, the letters $c_0, c_1, c_2, \ldots\ldots$ appear, they are supposed to be the coefficients in the expansion of $(1+x)^n$ where n is a positive integer, and the last of them will therefore be c_n.)

1. Find the middle term in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$.

2. If m and n are positive integers, shew that the coefficient of x^m in the expansion of $(1-x)^{-(n+1)}$ is equal to the coefficient of x^n in the expansion of $(1-x)^{-(m+1)}$.

3. Shew that, in the infinite series which gives a binomial expansion, the terms are sooner or later of the same sign or of alternate signs.

Illustrate each possibility.

4. Shew that

$$\frac{c_1}{c_0} + 2\frac{c_2}{c_1} + 3\frac{c_3}{c_2} + \ldots\ldots + n\frac{c_n}{c_{n-1}} = \frac{n(n+1)}{2}.$$

5. Find the coefficient of x^{n+r} in the expansions of $\frac{(1+x)^n}{1-x}$ and $\frac{(1+x)^n}{(1-x)^2}$.

6. Sum to infinity:

(1) $1 + \frac{1}{10} + \frac{1.3}{10.20} + \frac{1.3.5}{10.20.30} + \ldots\ldots$

(2) $\frac{1}{9} + \frac{1.4}{9.18} + \frac{1.4.7}{9.18.27} + \ldots\ldots$

7. Find the sum of the first $n+r$ coefficients in the expansion of $\frac{(1+x)^n}{1-x}$.

8. If n is a positive integer, shew that

$$1-\frac{n^2}{1^2}+\frac{n^2(n^2-1^2)}{1^2.2^2}-\ldots\ldots=0.$$

How many terms are there in this series?

9. Shew that

$$(c_0+c_1)(c_1+c_2)\ldots\ldots(c_{n-1}+c_n)=c_0c_1\ldots\ldots c_n.\frac{(n+1)^n}{n!}.$$

10. Find the value of the remainder after n terms in the expansions of $(1-x)^{-1}$ and $(1-x)^{-2}$.

11. Shew that

$$c_0+\frac{c_1}{2}+\frac{c_2}{3}+\ldots\ldots+\frac{c_n}{n+1}=\frac{2^{n+1}-1}{n+1}.$$

12. Find the coefficient of $x^5y^7z^{13}$ in the expansion of $(x+y+z)^{25}$.

13. Shew that

$$c_1-2c_2+3c_3-\ldots\ldots+n(-1)^{n-1}c_n=0.$$

14. By treating $1-2x+3x^2$ as a binomial find the coefficient of x^4 in the expansion of $(1-2x+3x^2)^n$.

15. Shew that

$$c_0+2c_1+3c_2+\ldots\ldots+(n+1)c_n=2^n+n2^{n-1}.$$

16. Shew that

$$c_1^2+2c_2^2+\ldots\ldots+nc_n^2=(2n-1)!\div(n-1)!\,(n-1)!.$$

17. If n is an odd positive integer shew that the integral part of $(\sqrt{2}+1)^n$ is $(\sqrt{2}+1)^n-(\sqrt{2}-1)^n$.

18. Shew that

$$c_0^2-c_1^2+\ldots\ldots+(-1)^nc_n^2$$

is equal to 0 if n is odd and to $(-1)^{\frac{n}{2}}n!\div(\frac{1}{2}n)!\,(\frac{1}{2}n)!$ if n is even.

19. If n is a positive integer shew that the integral part of $(2+\sqrt{3})^n$ is an odd integer.

CHAPTER IX

THE EXPONENTIAL AND LOGARITHMIC SERIES

1. The Exponential Series. It is proposed to make a brief study of the infinite series

$$1+\frac{x}{1}+\frac{x^2}{1.2}+\ldots\ldots+\frac{x^r}{1.2\ldots\ldots r}+\ldots\ldots \qquad (1)$$

a series which, on account of its simple form, might very easily have suggested itself for examination. This series has a finite limit for all finite values of x, a fact which will be assumed. Thus for $x=1$ the series is

$$1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\ldots\ldots+\frac{1}{1.2.3\ldots\ldots r}+\ldots\ldots \qquad (2)$$

This last series is seen to be less than

$$1+1+\frac{1}{2}+\frac{1}{2.2}+\frac{1}{2.2.2}+\ldots\ldots+\frac{1}{2^{r-1}}+\ldots\ldots$$

which after the first term is an infinite geometrical progression with common ratio $\frac{1}{2}$ so that its sum is

$$1+\frac{1}{1-\frac{1}{2}} \text{ or } 3.$$

Thus the series (2) has a finite limit between 2 and 3. This limit can, by taking a sufficient number of terms, be found to any degree of accuracy but it cannot be computed exactly. Its value is denoted by e so that

$$e=1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\ldots\ldots \textit{ in inf.} \qquad (I)$$

Approximately $e=2{\cdot}7182818$

Denote the series (1) by $F(x)$, thus indicating that it is a function of x. Then putting for x the values m and n we have

$$F(m)=1+\frac{m}{1!}+\frac{m^2}{2!}+\ldots\ldots+\frac{m^r}{r!}+\ldots\ldots$$

$$F(n)=1+\frac{n}{1!}+\frac{n^2}{2!}+\ldots\ldots+\frac{n^r}{r!}+\ldots\ldots$$

Assuming that these series may be multiplied as if they were two polynomials we have

$F(m).F(n)=1+\frac{1}{1!}(m+n)+$ terms of higher dimensions in m and n.

In this product the term of r dimensions in m and n is

$$\frac{m^r}{r!}+\frac{m^{r-1}}{(r-1)!}\cdot\frac{n}{1!}+\frac{m^{r-2}}{(r-2)!}\cdot\frac{n^2}{2!}+\ldots\ldots+\frac{n^r}{r!}$$

which can be put in the form

$$\frac{1}{r!}\left[m^r+\frac{r}{1}m^{r-1}n+\frac{r(r-1)}{1.2}m^{r-2}n^2+\ldots\ldots+n^r\right]$$

which is equal to $\dfrac{(m+n)^r}{r!}$.

Then giving to r the values 1, 2, 3, ... we have

$$F(m).F(n)=1+\frac{(m+n)}{1!}+\frac{(m+n)^2}{2!}+\ldots\ldots+\frac{(m+n)^r}{r!}+\ldots\ldots$$

that is, the product is the result of putting $m+n$ in place of (x) in (1). Therefore,

$$F(m).F(n)=F(m+n). \qquad (II)$$

Let, now, x be a positive integer. Then, by repeated application of (*II*), it follows that

$$F(1).F(1)\ldots\ldots \text{ to } x \text{ factors} = F(1+1+\ldots\ldots \text{ to } x \text{ terms}).$$

$$\therefore \left\{F(1)\right\}^x=F(x) \qquad (III)$$

Next let $\frac{p}{q}$ be any positive fraction, p and q being integers. Then

$$F\left(\frac{p}{q}\right).F\left(\frac{p}{q}\right)\ldots\ldots \text{ to } q \text{ factors} = F\left(\frac{p}{q}+\frac{p}{q}+\ldots\ldots \text{ to } q \text{ terms}\right)$$

$\therefore \left\{F\left(\frac{p}{q}\right)\right\}^q=F(p)=\left\{F(1)\right\}^p$, by (*III*) since p is a positive integer.

Then extracting the qth root

$$F\left(\frac{p}{q}\right)=\left\{F(1)\right\}^{\frac{p}{q}} \qquad (IV)$$

Thus from (III) and (IV) it follows that for all positive integral or fractional values of x

$$\left\{F(1)\right\}^{x}=F(x).$$

Finally let x be a negative fraction or integer and equal to $-y$ so that y is positive. By (II) we have

$$F(-y)\,.\,F(y)=F(-y+y)=F(0) \text{ which is seen to be 1.}$$

$$\therefore\ F(-y)=\frac{1}{F(y)}=\frac{1}{\left\{F(1)\right\}^{y}}=\left\{F(1)\right\}^{-y}$$

Thus also, replacing $-y$ by x, we have when x is a negative integer or fraction

$$F(x)=\left\{F(1)\right\}^{x}$$

Hence for all positive or negative integral or fractional values of x,

$$\left\{F(1)\right\}^{x}=F(x).$$

But $F(1)=e(=2{\cdot}7182818$ approximately). Therefore

$$e^{x}=1+\frac{x}{1!}+\frac{x^2}{2!}+\ \ldots\ldots\ +\frac{x^r}{r!}+\ \ldots\ldots \qquad (V)$$

This is the **exponential theorem.**

2. **Logarithmic Theorem.** In (V) of the preceding article replace x by $x\log_e a$. Then, since $e^{x\log_e a}=\left\{e^{\log_e a}\right\}^{x}=a^{x}$ by the definition of logarithm, it follows that

$$a^{x}=1+\frac{x\log_e a}{1}+\frac{x^2.(\log_e a)^2}{1.2}+\ \ldots\ldots$$

In this put $a=1+y$.

$$\therefore\quad (1+y)^{x}=1+\frac{x\log_e(1+y)}{1}+\frac{x^2.\left\{\log_e(1+y)\right\}^2}{1.2}+\ \ldots\ldots$$

Let y be numerically less than 1 so that we may expand $(1+y)^x$ by the binomial theorem. Then

$$(1+y)^x = 1 + x.y + \frac{x(x-1)}{1.2}y^2 + \ldots + \frac{x(x-1)\ldots(x-r+1)}{1.2\ldots r}y^r + \ldots$$

Then equating the coefficient of x in the two values of $(1+y)^x$, we have

$$\log_e(1+y) = \frac{y}{1} - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots.$$

This is valid for values of y numerically less than 1. Replacing y by x as the theorem is generally quoted in terms of x, we have for $|x| < 1$

$$\log_e(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots..$$

This is the **logarithmic theorem.**

The equality can be shewn to hold for the extreme value $x = 1$, but not for $x = -1$, neither side then having a meaning as log 0 is excluded from the definition of logarithm and the series $1 + \frac{1}{2} + \frac{1}{3} + \ldots.$ does not admit the idea of *sum*. Accordingly the theorem is to be stated thus:

For values of x such that $-1 < x \leqq 1$,

$$\log_e(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots.$$

If to x we give the value $-\frac{1}{2}$ we can find to any degree of accuracy the value of $\log_e 2$, and then, by putting $x = -\frac{1}{3}$, the value of $\log_e 3$, and so on. Or we might give to x first the value 1, then the value $\frac{1}{2}$, then $\frac{1}{3}$, and so on, to find in succession the values of $\log_e 2$, $\log_e 3, \ldots.$ If in either way these values be found, it will be seen that to attain a modest degree of accuracy a great many terms have to be taken. A series will now be obtained by means of which the work of computation will be greatly reduced.

In the series for $\log_e(1+x)$, put $x = \dfrac{1}{2n+1}$ where n is any positive integer, so that $\dfrac{1}{2n+1} < 1$. Then since $1 + \dfrac{1}{2n+1} = \dfrac{2(n+1)}{2n+1}$,

$$\log_e \frac{2(n+1)}{2n+1} = \frac{1}{1}\cdot\frac{1}{2n+1} - \frac{1}{2}\cdot\left(\frac{1}{2n+1}\right)^2 + \frac{1}{3}\cdot\left(\frac{1}{2n+1}\right)^3 - \frac{1}{4}\cdot\left(\frac{1}{2n+1}\right)^4 + \ldots$$

In like manner by putting $x = -\frac{1}{2n+1}$,

$$\log_e \frac{2n}{2n+1} = -\frac{1}{1} \cdot \frac{1}{2n+1} - \frac{1}{2} \cdot \left(\frac{1}{2n+1}\right)^2 - \frac{1}{3} \cdot \left(\frac{1}{2n+1}\right)^3 - \frac{1}{4} \cdot \left(\frac{1}{2n+1}\right)^4 - \dots$$

Therefore by subtraction, since the difference of the logarithms of two numbers is equal to the logarithm of their quotient,

$$\log_e \left(\frac{n+1}{n}\right) = 2\left[\frac{1}{1} \cdot \frac{1}{2n+1} + \frac{1}{3} \cdot \left(\frac{1}{2n+1}\right)^3 + \frac{1}{5} \cdot \left(\frac{1}{2n+1}\right)^5 + \dots\right]$$

In this last series the special virtue lies in the fact that soon high powers of a number less than 1 are reached, and it is to be expected that comparatively few terms need be taken to give a fairly good result, in other words, that the series is *rapidly convergent.*

It has been supposed that n is a positive integer in view of the fact that in tables it suffices to give the logarithms of the successive integers. If in the relation last found $n = 1$, the logarithm of 2 is given and a good working value will be given by the first six or seven terms. Then if $n = 2$ the value of $\log_e 3 - \log_e 2$ is given and thus $\log_e 3$ is known. Next if $n = 3$ the value of $\log_e 4 - \log_e 3$ is given and therefore also of $\log_e 4$ which can be checked since $\log_e 4 = 2 \log_e e$. So for $n = 4, 5, 6, \dots$.

The computations of the logarithms of the integers 2, 3, 4, ..., up to 10 at least, should be made. It would be seen that for the later of these integers very few terms of the series need be taken to give fairly good values. It is not difficult to reach a sort of estimate of the error made in taking a number of terms in place of the complete series. Suppose r terms taken; the series following the rth term of the series—say the *remainder after r terms* and denote it by R_r—is

$$2\left[\frac{1}{2r+1} \cdot \left(\frac{1}{2n+1}\right)^{2r+1} + \frac{1}{2r+3} \cdot \left(\frac{1}{2n+1}\right)^{2r+3} + \frac{1}{2r+5} \cdot \left(\frac{1}{2n+1}\right)^{2r+5} + \dots\right]$$

a series which has a sum, not however obtainable.
Consider now the infinite geometrical series

$$2\left[\frac{1}{2r+1} \cdot \left(\frac{1}{2n+1}\right)^{2r+1} + \frac{1}{2r+1} \cdot \left(\frac{1}{2n+1}\right)^{2r+3} + \frac{1}{2r+1} \cdot \left(\frac{1}{2n+1}\right)^{2r+5} + \dots\right]$$

with common ratio $(\frac{1}{2n+1})^2$, a number less than 1. This series has its terms after the first all less than the corresponding terms of the earlier series, and its sum is

$$2 \cdot \frac{1}{2r+1} \cdot \left(\frac{1}{2n+1}\right)^{2r+1} \cdot \frac{1}{1-\left(\frac{1}{2n+1}\right)^2} \text{ or } \frac{1}{2} \cdot \frac{1}{2r+1} \cdot \frac{1}{n(n+1)} \cdot \left(\frac{1}{2n+1}\right)^{2r-1}$$

Hence

$$R_r < \frac{1}{2} \cdot \frac{1}{2r+1} \cdot \frac{1}{n(n+1)} \cdot \left(\frac{1}{2n+1}\right)^{2r-1}$$

This last number is not then the value of R_r but is a *limit* within or below which R_r must be, like the 0·1, say, when a measurement is said to be *correct* to 0·1 inches. The limit for R_r is formidable looking, yet in actual computation it turns out, as will be seen, that it is not needed until it is virtually in hand. Suppose in the series for $\log_e (n+1) - \log_e n$ that $n = 5$, so that the value of $\log_e 6 - \log_e 5$ is to be found, and that the result is to be correct to six places of decimals.

$$\text{Formula gives: } \log_e 6 - \log_e 5 = 2\left[\frac{1}{1} \cdot \frac{1}{11} + \frac{1}{3} \cdot \frac{1}{11^3} + \frac{1}{5} \cdot \frac{1}{11^5} \cdot + \ldots\ldots\right]$$

$\frac{1}{11} = 0·09090909\ldots$: $\frac{1}{1} \cdot \frac{1}{11} = 0·09090909\ldots$

$\frac{1}{11^2} = 0·00826446\ldots$:

$\frac{1}{11^3} = 0·00075131\ldots$: $\frac{1}{3} \cdot \frac{1}{11^3} = 0·00025043\ldots$

$\frac{1}{11^4} = 0·00006830\ldots$:

$\frac{1}{11^5} = 0·00000620\ldots$: $\frac{1}{5} \cdot \frac{1}{11^5} = 0·00000124\ldots$

$0·09118076\ldots$

The divisions are carried to eight places, as probably sufficient to provide for accuracy to the sixth place. When the fifth division by 11 is made and therefore three terms of the series we are using

provided for, it is easy to see that the limit for the remainder after three terms is small enough to ensure that in taking three terms we shall have a sufficiently good result; for

$$R_3 < \frac{1}{2} \cdot \frac{1}{7} \cdot \frac{1}{5 \cdot 6} \cdot \frac{1}{11^5},$$

whence it follows, from the value of $\frac{1}{11^5}$ already found, that $R_3 < 0 \cdot 00000002$. Three terms will therefore give a satisfactory result. There is also another source of error, for in taking only eight places in evaluating each of the first three terms we have results too small. However, each of the results to be added differs from the actual value by less than 1 in the eighth place. Hence in taking

$$\log_e 6 - \log_e 5 = 0 \cdot 09116076 \times 2 \text{ or } 0 \cdot 18232152,$$

we have a result which differs from and is less than the actual value by less than $0 \cdot 00000006 + 0 \cdot 00000002$. Hence we may write

$$\log_e 6 - \log_e 5 = 0 \cdot 182322.$$

The ordinary scale of notation being ten, there is a great advantage in having logarithms calculated to ten as base. The series that has been found gives results to the base e so that the problem of changing the base arises. From the definition of logarithm it is easy to show that

$$\log_{10} N = \frac{\log_e N}{\log_e 10}.$$

Hence if the logarithms of the successive integers, calculated to base e be divided by $\log_e 10$, there will result the logarithms of the successive integers to base 10. It is found that $\log_e 10 = 2 \cdot 3025851$, but as a multiplication is to be preferred to a division the value of $1 \div \log_e 10$ is calculated and found to be $0 \cdot 4342945$. This number is the *modulus* for changing (by multiplication) logarithms from base e to base 10.

EXERCISES

1. Shew that the remainder after ten terms of the series defining e is less than

$$\frac{1}{10\,!}\cdot\left[1+\frac{1}{10}+\frac{1}{100}+\ldots\right]$$

and therefore less than $\frac{1}{9}\cdot\frac{1}{9\,!}$.

2. Employ the result in Ex. 1, to obtain an approximation to the value of e, discussing the *closeness* of the approximation.

3. Shew that the remainder after n terms of the series for e is less than

$$\frac{1}{n-1}\cdot\frac{1}{(n-1)\,!}.$$

4. Find a series for $\frac{e+e^{-1}}{2}$ and for $\frac{e-e^{-1}}{2}$.

5. Find a series for $\frac{e^x+e^{-x}}{2}$ and for $\frac{e^x-e^{-x}}{2}$.

6. Find an approximate value for e^{-1} from the series giving this number and test the result by division of 1 by the known approximation to e.

7. Write down the series for e^2 and employ it to find the value of e^2 to five places of decimals.

Test the results by squaring the known approximation to e.

8. From the series for $\log_e\left(\frac{1+x}{1-x}\right)$, shew that

(i) $\log_e n=2\left\{\frac{1}{1}\cdot\frac{n-1}{n+1}+\frac{1}{3}\left(\frac{n-1}{n+1}\right)^3+\ldots\right\}$

(ii) $\log_e\frac{m}{n}=2.\left\{\frac{1}{1}\cdot\frac{m-n}{m+n}+\frac{1}{3}\cdot\left(\frac{m-n}{m+n}\right)^3+\ldots\right\}$

(iii) $\log_e(n+1)=2\ \log_e n-\log_e(n-1)-2\left\{\frac{1}{1}\cdot\frac{1}{2n^2-1}+\frac{1}{3}\cdot\frac{1}{(2n^2-1)^3}+\ldots\right\}$ m and n being positive.

9. In (iii) of Ex. 8, examine the results of substituting $n=4$, and $n=6$.

MISCELLANEOUS EXAMPLES

MISCELLANEOUS EXAMPLES

I

1. If $a^2 = y + z$, $b^2 = z + x$, $c^2 = x + y$, and $2s = a + b + c$, shew that

$$4s(s-a)(s-b)(s-c) = (yz + zx + xy).$$

2. Resolve into linear factors :

$$6x^2 - x - 15, \quad 6x^2 - 31xy + 35y^2,$$
$$x^2 + 8x + 9, \quad x^2 + 5xy + 3y^2,$$
$$2x^2 - 3x - 7, \quad 4x^2 - 11xy + 5y^2.$$

3. Shew by repeated division by 7 that

$$5194915 = 6.7^7 + 2.7^6 + 1.7^5 + 4.7^3 + 3.7^2 + 4.7 + 5$$

or, otherwise expressed, that 5194915 *in the scale of ten* is equal to 62104345 *in the scale of seven.*

Express

(1) 32974 in the scale of 5.

(2) 98573 in the scale of 9.

(3) 13827 in the scale of 2.

(4) 83141 in the scale of 13, employing t, e, z to denote 10, 11. 12.

4. Solve

$$(x+2)(x+3)(x+4)(x+5) = 1680.$$

5. If $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$, then

$$\frac{(x^2+y^2+z^2)(yz+zx+xy)}{(l^2+m^2+n^2)(mn+nl+lm)} = \frac{x^4+y^4+z^4}{l^4+m^4+n^4}.$$

II

1. Construct a geometrical representation of each of the formulæ:

(1) $(a+b)^2 = a^2 + 2ab + b^2$;

(2) $(x+a)(x+b) = x^2 + (a+b)x + ab$;

(3) $(x+y)(x-y) = x^2 - y^2$;

(4) $(x+y)^2 + (x-y)^2 = 2(x^2+y^2)$.

Give to each formula a verbal statement.

2. Employ the factor theorem (*v.* page 17) to shew that for positive integral values of n,

(1) $x^n - y^n$ is divisible by $x - y$;

(2) $x^{2n} - y^{2n}$ is divisible by $x + y$;

(3) $x^{2n+1} - y^{2n+1}$ is not divisible by $x + y$.

3. Perform the following computations in the scale of 7:

(1) $3652 + 2153 + 3154 + 1421 + 243$.

(2) $5136 - 3654$

(3) 21354×4.

(4) $156341 \div 25$.

4. Solve

$$(4x^2 + 16x + 15)(x^2 + 7x + 12) = 700.$$

5. Solve the simultaneous set of equations

$$2x - 3y = 3,$$
$$3x - 2y = 7.$$

Also for each find y in terms of x and represent y graphically.

III

1. If $2s = a + b + c$ and $2r^2 = a^2 + b^2 + c^2$, shew that

$$(r^2 - b^2)(r^2 - c^2) + (r^2 - c^2)(r^2 - a^2) + (r^2 - a^2)(r^2 - b^2)$$
$$= 4s(s-a)(s-b)(s-c).$$

2. Factor:

$$3x^2 + 11xy + 10y^2 + 17x + 31y + 24,$$
$$6x^2 - 12y^2 - 28z^2 + 37yz - 2zx - xy.$$

3. Perform the following computations in the scale of 13 where t, e, z are the digits denoting ten, eleven, twelve:

(1) $5e39t + 4372z + 1z84e + 57t31$.

(2) $53{\cdot}4e + 0{\cdot}3z5 + 5{\cdot}247 + 3{\cdot}zte$.

(3) $7123t5 - 4z35e9$.

(4) $53{\cdot}47 \times 23$; $21{\cdot}42 \times 2{\cdot}3$.

(5) $5934t \div 9$; $120{\cdot}0z \div 2{\cdot}e$.

4. Solve

$$(4x^2 + 16x + 15)(x^2 + 5x + 6) = 420.$$

5. If

$$z - a : z - b :: z - c : z - d$$

and

$$a^2 + ad + d^2 = b^2 + bc + c^2,$$

then

$$z = a + b + c + d.$$

IV

1. If A, B, C are three points on a straight line, and P any other point on the line, then

(i) $PA.BC + PB.CA + PC.AB = 0$,

(ii) $PA^2.BC + PB^2.CA + PC^2.AB = -BC.CA.AB$,

it being understood that lengths measured in opposite directions in the line are affected with unlike signs.

2. Employ the factor theorem to shew that, for positive integral values of n,

(1) $x^{2n+1}+y^{2n+1}$ is divisible by $x+y$,

(2) $x^{2n} \quad +y^{2n}$ is not divisible by $x+y$,

(3) $x^{n} \quad +y^{n}$ is not divisible by $x-y$.

3. Express 58437, given in the scale of 10, in the scale of 7, transform the result to the scale of 13 and then transform the result to the scale of 10.

4. Solve

$$24x^4+22x^3-47x^2+22x+24=0.$$

5. Shew that the two equations

$$2x+3y=x+4y+3$$

$$3x+5y=x+7y+7$$

are *inconsistent*, *i.e.*, cannot be satisfied by the same set of values of x and y.

In each case find y in terms of x and represent y graphically.

V

1. Verify the following expressions for the cube of the trinomial $a+b+c$:

(1) $a^3+b^3+c^3+3a^2(b+c)+3b^2(c+a)+3c^2(a+b)+6abc$;

(2) $a^3+b^3+c^3+3(b+c)(c+a)(a+b)$;

(3) $a^3+b^3+c^3+3(a+b+c)(bc+ca+ab)-3abc.$

Give to each of the corresponding formulæ a verbal statement.

2. Factor

$$1-2ax-(c-a^2)x^2+acx^3;$$

$$a^2x^3-b^2xy^2-a^2cx^2+b^2cy^2;$$

$$a^2(a+b+c+d)+abc+bcd+cda+dab.$$

3. Find the square roots of the following numbers in the scale of 7:

$$562,\ 4621,\ 15462,\ 2050{\cdot}21.$$

4. Solve

$$(x+3)^4+(x+5)^4=3026.$$

5. If

$$x(b-c)+y(c-a)+z(a-b)=0,$$

shew that

$$(1)\ \frac{bz-cy}{b-c}=\frac{cx-az}{c-a}=\frac{ay-bx}{a-b};$$

$$(2)\ \frac{y-z}{b-c}=\frac{z-x}{c-a}=\frac{x-y}{a-b}.$$

VI

1. If A, B, C, P, Q are any five collinear points, then

$$\frac{\text{AP.AQ}}{\text{AB.AC}}+\frac{\text{BP.BQ}}{\text{BC.BA}}+\frac{\text{CP.CQ}}{\text{CA.CB}}=1.$$

2. If $f(x)$ is any polynomial, then

$$f(x)-f(a)$$

is always divisible by $x-a$.

3. In what system of numeration will the number 435 (base 10) be written 861?

4. Solve

$$x^2-y^2=16,$$
$$xy=15.$$

5. Shew that the following two equations

$$2x-\ 3y=0,$$
$$8x-13y=0,$$

are inconsistent unless x and y are both zero.

Represent y graphically in each case.

VII

1. If $x+y=p$, $xy=q$, find x^2+y^2, x^3+y^3, x^4+y^4, x^5+y^5 in terms of p and q.

2. Factor:

$$x^4+x^2y^2+y^4;$$

$$x^4-2abx^2-a^4-a^2b^2-b^4;$$

$$(1+y)^2-2x^2(1+y^2)+x^4(1-y)^2;$$

$$(bx+ay)^4-4abxy(bx+ay)^2-(a^2x^2-b^2y^2)^2.$$

3. Shew that any number written in the scale of ten is divisible by 9 if the sum of the digits is divisible by 9.

4. Solve

$$x^2+y^2+3(x-y)=44,$$

$$x+y=\ 9.$$

5. If

$$\frac{a^2-bc}{a-abc}=\frac{b^2-ac}{b-abc}=\frac{c^2-ab}{a-abc},$$

then

$$a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

VIII

1. If A, B, C, D, P are any five collinear points, then

$$\frac{\text{AP}}{\text{AB.AC.AD}}+\frac{\text{BP}}{\text{BC.BD.BA}}+\frac{\text{CP}}{\text{CD.CA.CB}}+\frac{\text{DP}}{\text{DA.DB.DC}}=0.$$

2. A polynomial in x is divisible by $x-1$ if the sum of its coefficients is zero, and divisible by $x+1$ if the sum of the coefficients of the odd powers of x is equal to the sum of the coefficients of the even powers.

3. Shew that any number written in the scale r is divisible by $r-1$ if the sum of its digits is divisible by $r-1$.

4. Solve

$$6x^2 - 17xy + 12y^2 = 0,$$
$$x^2 + 3xy - 5y = 17.$$

5. Shew that the following three equations are inconsistent :

$$2x + 5y = 24,$$
$$7x - 3y = 9,$$
$$4x + 11y = 45.$$

IX

1. If $x^2 - px + 1 = 0$, $y^2 - qy + 1 = 0$, $z^2 - rz + 1 = 0$, then

$$(x^3 + y^3 + z^3) + (x^{-3} + y^{-3} + z^{-3})$$
$$= p(p^2 - 3) + q(q^2 - 3) + r(r^2 - 3).$$

2. Factor :

$$2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4,$$
$$4(ad + bc)^2 - (a^2 - b^2 - c^2 + d^2)^2.$$

3. Shew that any number written in the scale of ten is divisible by 11 if the difference between the sum of the digits in the odd places and the sum of those in the even places is divisible by 11.

4. Solve :

$$xy(x + y) = \frac{5}{36},$$
$$\frac{1}{x} + \frac{1}{y} = 5.$$

5. If

$$\frac{b - c}{2y + 3z} = \frac{2(c - a)}{4z + 5x} = \frac{3(a - b)}{7x - y},$$

then

$$29x + 10y + 30z = 0.$$

X

1. If a, b, c measure the sides of a triangle ABC, and if p measures the perpendicular from the vertex A to the side a, find p in terms of a, b, c and obtain the expression

$$\sqrt{s(s-a)\,(s-b)\,(s-c)}$$

for the area of a triangle.

2. If

$$x^n + p_1 x^{n-1} + \ldots + p_n,$$

where $p_1, p_2, \ldots, p_n$ are positive or negative integers, is divisible by $x - a$, where a is an integer, shew that a is a factor of p_n.

3. Shew that any number written in the scale of r is divisible by $r + 1$ if the sum of the digits in the odd places is equal to the sum of the digits in the even places.

4. Solve:

$$\frac{x+y+6}{2} - \frac{1}{x+y-5} = 4,$$

$$xy = 3.$$

5. The three equations

$$\begin{aligned} 2x - 3y + 5z &= 0, \\ x + 2y - \ z &= 0, \\ 3x + \ y - 7z &= 0, \end{aligned}$$

are inconsistent unless

$$x = y = z = 0.$$

XI

1. If $a^2 + b^2 + c^2 = bc + ca + ab$, then $a = b = c$.

2. By means of the remainder theorem shew that $a - b$ is a factor of

$$a^2(b-c) + b^2(c-a) + c^2(a-b),$$

and write the given expression as a product of factors.

Factor :

(1) $bc(b-c)+ca(c-a)+ab(a-b)$;

(2) $(x+y+z)^3-x^3-y^3-z^3$;

(3) $(y-z)^3+(z-x)^3+(x-y)^3$.

3. Express $\frac{5}{8}$ as a decimal (or *radix*) fraction, explaining the process.

In like manner shew that $\frac{23}{36}$, given in the scale of 10, is equal to $\frac{7}{12}+\frac{8}{12^2}$, *i.e.*, to the *radix fraction* 0·78 in the scale of 12.

Obtain this result also by changing the numerator and denominator of the given fraction to the scale of 12, and performing the indicated division.

4. Solve

$$12x^5-8x^4-45x^3+45x^2+8x-12=0.$$

5. If

$$\frac{m}{x}=\frac{n}{y}=\frac{p}{z} \text{ and } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

then

$$\frac{m^2}{a^2}+\frac{n^2}{b^2}+\frac{p^2}{c^2}=\frac{m^2+n^2+p^2}{x^2+y^2+z^2}.$$

XII

1. Distinguish between an identical equation and a conditional equation, and explain the statement: *Every conditional equation is a hypothetical identity* (Chrystal, *Text Book of Algebra*, Part I, page 286).

2. By means of the remainder theorem shew that $a+b+c$ is a factor of

$$a^3+b^3+c^3-3abc,$$

and, noting the degree and symmetry of the expression, obtain without actual division the formula

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab).$$

Also obtain this result by adding to a^3+b^3 in the given expression to complete the cube of $a+b$.

3. Shew that

(1) $\frac{2}{3} = 0{\cdot}4$ in the scale of 6;

(2) $\frac{5}{8} = 0{\cdot}68$ in the scale of 12;

(3) $\frac{5}{7} = 0{\cdot}\dot{7}9\dot{4}$ in the scale of 11.

Does the question, whether the given vulgar fractions are in the scale of 10, arise?

4. Solve

$$x^5 + 1 = 0.$$

5. (*a*) If x satisfies the two equations

$$ax + b = 0, \quad a'x + b' = 0,$$

explain why it is to be concluded that there must exist some relation among the quantities a, b, a', b'.

Shew that the relation is

$$ab' - a'b = 0,$$

or, in a notation much in use,

$$(ab') = 0.$$

(*b*) If the equations

$$ax + by = 0, \quad a'x + b'y = 0,$$

are simultaneously true, and if x, y are not zero, shew that

$$ab' - a'b = 0.$$

XIII

1. Shew that, if

$$(x^2 + y^2 + z^2)(a^2 + b^2 + c^2) = (ax + by + cz)^2,$$

then

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$$

2. Shew that $b - c$ is a factor of

$$a(b - c)^3 + b(c - a)^3 + c(a - b)^3.$$

Infer two other factors, and, noting the degree and symmetry of the given expression, determine the remaining factor.

Factor:

(1) $a^3(b-c)+b^3(c-a)+c^3(a-b)$;

(2) $(x+y+z)^4-(y+z)^4-(z+x)^4-(x+y)^4+x^4+y^4+z^4$.

3. Express $\frac{15}{16}$, $\frac{17}{18}$, $\frac{16}{19}$ as radix fractions in the scale of 12, supposing the given fractions to have their terms written (i) in the scale of 10, (ii) in the scale of 12.

4. Solve

$$12x^4-28x^3-9x^2+28x+12=0.$$

5. If

$$\frac{xyz}{y+z}-x^2=\frac{xyz}{z+x}-y^2$$

then each of these expressions is equal to $yz+zx+xy$, and to

$$\frac{xyz}{x+y}-z^2,$$

unless $x=y$.

XIV

1. Form the equation whose roots are the products by r of the roots of the equation

$$ax^2+2bx+c=0.$$

2. Factor:

(1) $b^2c^2(b-c)+c^2a^2(a-b)+a^2b^2(a-b)$;

(2) $a^4(b-c)+b^4(c-a)+c^4(a-b)$.

3. In the scale of 12 what vulgar fractions will give rise to terminating radix fractions?

4. Solve

$$8x^6-42x^5+21x^4+84x^3-21x^2-42x-8=0.$$

5. If the three equations

$$a\,x+b\,y+c\,=0,$$
$$a'\,x+b'\,y+c'\,=0,$$
$$a''x+b''y+c''=0,$$

are satisfied by the same values of x, y, shew that there must exist a relation among the known quantities appearing in the equations, and that this relation is

$$a(b'c''-b''c')+b(c'a''-c''a')+c(a'b''-a''b')=0$$

or

$$a(b'c'')+b(c'a'')+c(a'b'')=0$$

or, in a convenient notation,

$$(ab'c'')=0.$$

This relation, as also that found in Ex. 5, xii (p. 132), shewn to exist if the unknowns satisfy a given set of equations, is called the *eliminant* of the given set of equations.

The following equations

$$2x+\ \ 3y-21=0,$$
$$7x+11y-76=0,$$
$$13x-\ \ 5y-15=0,$$

are not *consistent.*

XV

1. If a, b, c, x, y, z are real quantities and

$$(a+b+c)^2=3\,(bc+ca+ab-x^2-y^2-z^2)$$

shew that $a=b=c$ and that $x=y=z=0$.

2. Factor:

(1) $(a+b+c)\,(bc+ca+ab)-abc$;

(2) $(b-c)\,(b+c)^3+(c-a)\,(c+a)^3+(a-b)\,(a+b)^3$;

(3) $(x+y+z)^5-(y+z-x)^5-(z+x-y)^5-(x+y-z)^5$.

3. Shew that any integer can be written in the form

$$a_0.1 + a_1.1.2 + a_2.1.2.3 + a_3.1.2.3.4 + \ldots\ldots$$

when $a_0 = 0$ or 1 and $a_1, a_2, a_3 \ldots\ldots$ are integers (zero included) less respectively than 2, 3, 4, ..., and that any fraction can be written in the form

$$\frac{a_0}{1} + \frac{a_1}{1.2} + \frac{a_2}{1.2.3} + \frac{a_3}{1.2.3.4} + \ldots\ldots$$

where a_0 is zero or an integer and $a_1, a_2, a_3 \ldots\ldots$ are integers (zero included) less respectively than 2, 3, 4, ...

4. Solve

$$\sqrt{3x^2 - 2x + 9} + (3x+1)(x-1) \quad = 46.$$

Discuss the question whether there are four roots or two.

5. If

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{z}, \text{ and also } \frac{y^2 - z^2}{b - c} = \frac{yz}{x},$$

then will

$$\frac{z^2 - x^2}{c - a} = \frac{zx}{y}.$$

XVI

1. The numerically greater root of

$$ax^2 - bx + c = 0$$

has the same sign as $\frac{b}{a}$ and the numerically less root the same sign as $\frac{b}{c}$, the roots being supposed to be real.

2. Shew that if $x^{n-1} - a^{n-1}$ is divisible by $x - a$, so also is $x^n - a^n$.

From this and the fact that $x - a$ or $x^2 - a^2$ is divisible by $x - a$, shew that, for all integral values of n, the expression $x^n - a^n$ is divisible by $x - a$.

3. The weight of a cubic block of lead varies as the cube of its edge. When the edge is 3 and the weight is 36. Find the consequent relation between the weight of a cubic block of lead and its surface.

4. Solve

$$\frac{1}{x}+\frac{1}{y}=\frac{x+y}{12}=\frac{7}{x+y+5}.$$

5. Shew that if the three equations

$$a\,x+b\,y+c\,z=0,$$
$$a'\,x+b'\,y+c'\,z=0,$$
$$a''x+b''y+c''z=0,$$

are satisfied by other than zero values of x, y, z, then there must exist among the coefficients the relation expressed by

$$(ab'c'')=0$$

As in Ex. 5, xiv (p. 134), this result is called the *eliminant* of the three homogeneous equations in three unknowns.

If x, y, z are not all zero the three equations

$$5x-2y-z=0,$$
$$x+6y-5z=0,$$
$$8x-y-3z=0,$$

are inconsistent.

XVII

1. If

$$(y-z)^2+(z-x)^2+(x-y)^2$$
$$=(y+z-2x)^2+(z+x-2y)^2+(x+y-2z)^2$$

and x, y, z are real then $x=y=z$.

2. If $m+in$, where i denotes $\sqrt{-1}$, is a root of the equation $ax^2+2bx+c=0$ where a, b, c are real, then $m-in$ is the other root.

3. Sum

$$(1)\ 1-5+9-13+\ldots\ldots \text{ to } 2n \text{ terms};$$

$$(2)\ 1-9+17-25+\ldots\ldots \text{ to } 2n+1 \text{ terms};$$

and obtain a formula which will give the sum of n terms of the series

$$1-3+5-7+\ldots\ldots$$

whether n be odd or even.

4. Solve

$$x+y+xy=11,$$
$$x^2y+xy^2=30.$$

5. If

$$\frac{x-\dfrac{yz}{x}}{1-yz}=\frac{y-\dfrac{zx}{y}}{1-zx},$$

and x and y are unequal, then will each member of this equation be equal to

$$\frac{z-\dfrac{xy}{z}}{1-xy}.$$

XVIII

1. Shew that the roots of $x^2+px+q=0$ will be rational if $p=k+\dfrac{q}{k}$, where p, q, k are any rational quantities.

2. Assuming that $x^{2n-1}+a^{2n-1}$ is divisible by $x+a$ shew that $x^{2n+1}+a^{2n+1}$ must be divisible by $x+a$.

From this and the fact that x^3+a^3 is divisible by $x+a$ what inference is to be made?

NOTE:—In this example, as in Ex. 2, xvi (p. 135), a general theorem is established by observing it to exist or be valid in some one particular case (a *first case*) and shewing that if it is valid in *any one case* it must be valid in the *next case*. It then follows that the theorem is true for

all cases, and it is to the totality of all such possible succeeding cases that the generality of the theorem refers. In the examples considered all cases are given by the successive positive integral values of n. This method of proof is known as *mathematical induction* or *complete induction*. There are many theorems which to be established by this method must be known to be valid in more than one particular case.

3. A variable y is known to vary as the sum of two quantities, one of which varies as x and the other as x^2. If when x is 1, y is 3 and when x is 2, y is 8 find the relation between the variables, and the value of y when x is 3.

Construct a graphic representation of y.

4. Solve

$$xy = 6,$$

$$3\frac{x}{y} + 2\frac{y}{x} = 5.$$

5. If the equations

$$ax + b = 0,$$

$$a'x + b' = 0,$$

are satisfied by the same value of x then identically

$$ax + b = l(a'x + b')$$

where l is some constant; and if the equations

$$a\,x + b\,y + c = 0,$$

$$a'x + b'y + c' = 0,$$

$$a''x + b''y + c'' = 0,$$

are satisfied by the same values of x, y, then identically

$$a''x + b''y + c''z = l(ax + by + cz) + l'(a'x + b'y + c'z)$$

where l, l' are certain constants.

XIX

1. If

$$l^2 + m^2 + n^2 = 1,$$

$$l'^2 + m'^2 + n'^2 = 1,$$

$$ll' + mm' + nn' = 1,$$

shew that

$$l = l',\ m = m',\ n = n'.$$

2. If $m + in$, where i denotes $\sqrt{-1}$, is a root of the equation

$$ax^3 + 3bx^2 + 3cx + d = 0$$

where a, b, c, d are real, then $m - in$ is also a root.

If one root of the equation

$$3x^3 - 13x^2 + 20x - 14 = 0$$

is $1 + i$ find the other roots.

3. Shew that, whatever positive integer r may be, the sum of the first r terms of the series

$$1 + 3 + 5 + \ldots\ldots$$

is one-third the sum of the next r terms. Prove also that this is virtually the only A.P. which has this property.

4. Solve

$$xy(x + y) = 30,$$

$$x^3 + y^3 = 35.$$

5. If

$$lx + my + nz = 0,$$

$$ax^2 + by^2 + cz^2 = 0,$$

find the ratios $x : y : z$.

XX

1. Solve

$$\sqrt{x+1}+\sqrt{x-1}=1.$$

and explain how it comes that the root obtained by rationalizing the equation does not satisfy the equation as written.

2. In the case of the series

$$1+3+5+7+\ldots\ldots$$

it is seen that the sum of 2 terms is 4 or 2^2, the sum of 3 terms is 9 or 3^2, the sum of 4 terms is 16 or 4^2. What theorem is suggested? Establish it by mathematical induction.

3. The strength of a rectangular beam of given length and material varies as its breadth and as the square of its depth. Compare the strengths of two such beams, the one of breadth 4 in. and depth 9 in., the other of breadth and depth each 6 in.

4. Solve

$$x+y=5,$$
$$x^4+y^4=257.$$

5. The equations

$$ax+b=0,$$
$$px^2+2qx+r=0,$$

are satisfied by the same value of x. Find the relation among the coefficients.

XXI

1. By means of a single equation express each of the following:—

(1) The value of x must be either a or b.

(2) The value of x is a, the value of y is b, and the value of z is c.

2. If $m+in$ is a root of the equation

$$ax^4+4bx^3+6cx^2+4dx+e=0$$

where a, b, c, d, e are real then $m-in$ is also a root.

If $2+i$ is a root of the equation

$$x^4-7x^3+18x^2-19x+5=0$$

find all the remaining roots.

3. Find how many terms of the series $15+21+27+\ldots\ldots$ must be taken in order to yield 663 as sum.

Account for the negative root.

4. Solve

$$\frac{yz}{x}=a,\ \frac{zx}{y}=b,\ \frac{xy}{z}=c.$$

5. If

$$ay+bx=a,\ by-ax=b,$$

then

$$x^2+y^2=1.$$

XXII

1. Shew that if two quadratics in two unknowns have a common factor in the parts involving the unknowns, the set reduces to a simple and a quadratic equation.

Ex.

$$\begin{aligned} x^2+xy-2y^2+x+2y&=66, \\ 3x^2-4xy+y^2+3x-y&=144. \end{aligned}$$

2. Shew by mathematical induction that the sum of n terms of the series

$$1+2+3+\ldots\ldots$$

is $\dfrac{n(n+1)}{2}$.

3. If

$$a+\sqrt{b}=c+\sqrt{d}$$

where $\sqrt{b}$, $\sqrt{d}$ are actual surds, shew that

$$a=c,\ b=d;$$

and if

$$a+ib=c+id,$$

shew that

$$a=c,\ b=d,$$

it being supposed that a, b, c, d are all real.

4. If

$$\frac{x-a}{b}+\frac{y-b}{c}+\frac{z-c}{a}=0,$$

$$\frac{x-b}{c}+\frac{y-c}{a}+\frac{z-a}{b}=0,$$

$$\frac{x-c}{a}+\frac{y-a}{b}+\frac{z-b}{c}=0,$$

then

$$x=y=z=\frac{ab^2+bc^2+ca^2}{ab+bc+ca}.$$

5. Eliminate x from the equations, supposed simultaneously existent,

$$a\,x^2+2b\,x+c\ =0,$$

$$a'x^2+2b'x+c'=0.$$

XXIII

1. Determine the numerical values of a, b, c, in order that the equation

$$a(x+2)\ (x+3)+b(x+3)\ (x+1)+c(x+1)\ (x+2)=4x+6$$

may be an identity.

Express as the sum of three fractions with constant numerators and with denominators $x+1$, $x+2$, $x+3$, the fraction

$$\frac{4x+6}{(x+1)\,(x+2)\,(x+3)}.$$

The given fraction is said to be resolved into *partial fractions.*

2. Shew that

$$x^3 - y^3 = (x - y)(x - \omega y)(x - \omega^2 y)$$

where ω denotes one of the imaginary cube roots of unity.

3. The sum of ten terms of a certain arithmetical progression is three times the sum of five terms; shew that the first term equals three times the common difference.

4. Solve

$$\begin{aligned} 2x - 3y + z &= 0, \\ x + 2y - 2z &= 0, \\ x^2 + 2y^2 - z^2 &= x + 4y - z. \end{aligned}$$

5. The equations

$$\begin{aligned} a x^2 + 2b x + c &= 0, \\ a' x^2 + 2b' x + c' &= 0, \end{aligned}$$

have two roots in common; find the relations that must exist among the coefficients.

XXIV

1. Assuming the theorem that every algebraic equation in one unknown (*i.e.*, every equation of the form $f(x) = 0$, where $f(x)$ is a polynomial in x) has at least one root, shew that every cubic equation has three and only three roots.

2. Shew by mathematical induction that the sum of n terms of the series

$$1^2 + 2^2 + 3^2 + \ldots\ldots$$

is $\dfrac{n(n+1)(2n+1)}{6}$.

3. Shew how to find $\sqrt{(p + \sqrt{q})}$, indicating the condition that the root may be of simpler form.

Ex. $\sqrt{(8 + \sqrt{60})}$, $\sqrt{(11 - \sqrt{120})}$, $\sqrt{(8 + \sqrt{15})}$.

4. Solve

$$\frac{x+y}{3}=\frac{y+z}{4}=\frac{z+x}{5},$$

$$6(yz+zx+xy)=11(x+y+z).$$

5. Eliminate x from the equations

$$ax^3+3bx+c=0,$$

$$a'x^3+3b'x+c'=0.$$

XXV

1. Express as a sum of partial fractions each of the following:

$$\frac{4x^2-28x+54}{(x-3)(x-4)(x-6)},\quad \frac{2x^2+7x-5}{x^2+2x-3},\quad \frac{1}{(x-a)(x-b)(x-c)}.$$

2. Shew that

$$(a+\omega^2 b+\omega c)^3-(a+\omega b+\omega^2 c)^3=3\sqrt{-3}(b-c)(c-a)(a-b)$$

where ω is an imaginary cube root of unity.

3. The first term of an arithmetical progression is 1 and the number of terms is even. If the sum of the odd terms is 287, and the sum of the even terms is 329, find the series.

4. Solve

$$b^2z+c^2y=c^2x+a^2z=a^2y+b^2x=xyz.$$

5. If

$$a=\frac{x-y}{x+y},\quad b=\frac{y-z}{y+z},\quad c=\frac{z-x}{z+x},$$

prove that

$$\frac{1+a}{1-a}\cdot\frac{1+b}{1-b}\cdot\frac{1+c}{1-c}=1.$$

XXVI

1. If α, β, γ are the three roots of the equation

$$ax^3 + 3bx^2 + 3cx + d = 0,$$

shew that

$$\alpha + \beta + \gamma = -\frac{3b}{a}; \quad \beta\gamma + \gamma\alpha + \alpha\beta = \frac{3c}{a}; \quad \alpha\beta\gamma = -\frac{d}{a}.$$

2. Shew by mathematical induction that the sum of n terms of the series

$$1 + r + r^2 + \ldots\ldots$$

is $\dfrac{1 - r^n}{1 - r}$.

3. Find a simpler expression for

$$\sqrt{(5 + 12\sqrt{-1})}, \quad \sqrt{(7 - 24\sqrt{-1})}.$$

4. Solve

$$(1 + x^2)(1 + y^2) = 2(xy - 1)^2, \quad x + y = 5.$$

5. Eliminate x from

$$a x^3 + b x^2 + c = 0,$$

$$a'x^3 + b'x^2 + c' = 0.$$

XXVII

1. Shew that

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$$

$$= (ac - bd)^2 + (ad + bc)^2.$$

Hence, shew that the product of two integers each of which is the sum of the squares of two integers can always be written as the sum of the squares of two integers in at least two ways.

Ex. $41 (= 4^2 + 5^2)$, $113 (= 7^2 + 8^2)$.

Is the converse true?

2. Shew that

$$a^3+b^3+c^3-3abc$$
$$=(a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)$$

where ω denotes one of the imaginary cube roots of unity.

3. Any odd integer may be expressed as the difference of two square integers.

4. Solve

$$x^2+y^2-x-y=78,$$
$$x+y+xy=39.$$

5. Eliminate l, m from the equations $lx+my=a$, $mx-ly=b$, $l^2+m^2=1$.

XXVIII

1. If α, β, γ are the roots of the equation

$$x^3+px+q=0,$$

construct the equation whose roots are α^2, β^2, γ^2.

2. Shew by mathematical induction that

$$1.2+2.3+3.4+\ldots\ldots \text{ to } n \text{ terms} = \tfrac{1}{3}n(n+1)(n+2).$$

3. If $\sqrt[3]{(a+\sqrt{b})}=x+\sqrt{y}$, then will $\sqrt[3]{(a-\sqrt{b})}=x-\sqrt{y}$.

4. Solve

$$x^2(y+2)+y^2(x+2)=56, \quad xy=6.$$

5. Eliminate x and y from the equations,

$$\frac{x}{a}+\frac{y}{b}=1, \quad \frac{x^2}{c^2}+\frac{y^2}{d^2}=1, \quad xy=k^2.$$

XXIX

1. Prove that

$$(x^2-y^2)(z^2-w^2)=(xz+yw)^2-(xw+yz)^2$$
$$=(xz-yw)^2-(xw-yz)^2.$$

Interpret as a theorem in integral numbers.

2. Shew that each of the expressions

$$(x^2+2yz)^3+(y^2+2zx)^3+(z^2+2xy)^3-3(x^2+2yz)(y^2+2zx)(z^2+2xy),$$
$$(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3-3(x^2-yz)(y^2-zx)(z^2-xy),$$

is an exact square.

3. If a is the first and d the last of a series of four numbers in harmonical progression, prove that the sum of the series of four terms is

$$\frac{2(a+d)(a^2+7ad+d^2)}{(a+2d)(2a+d)}.$$

4. Solve

$$x^3(y+3)+y^3(x+3)=183, \quad x+y=5.$$

5. If

$$x=\frac{b-c}{a}, \quad y=\frac{c-a}{b}, \quad z=\frac{a-b}{c}$$

then

$$xyz+x+y+z=0.$$

XXX

1. Construct the equation whose roots exceed the three roots of the equation

$$x^2+px+q=0$$

by h.

2. Shew by mathematical induction that

$$\frac{1}{1.2}+\frac{1}{2.3}+\ldots\ldots \text{ to } n \text{ terms } =\frac{n}{n+1}.$$

3. Simplify:

$$\frac{1}{\sqrt{7}+\sqrt{5}}, \quad \frac{1}{\sqrt{3}+\sqrt{5}+\sqrt{7}}.$$

4. Solve

$$x^4 - x^2y^2 + y^4 = 85,$$

$$xy(x^2 - y^2) = 18,$$

by putting $z = x^2 - y^2$, $w = xy$.

5. If

$$mx_1^2 + ny_1^2 = a^2,$$

$$mx_2^2 + ny_2^2 = a^2,$$

$$mx_1x_2 + ny_1y_2 = 0$$

then

$$x_1^2 + x_2^2 = \frac{a^2}{m}, \quad y_1^2 + y_2^2 = \frac{a^2}{n}.$$

XXXI

1. Shew that the product of n different integers, each of which is the sum of two square integers, may be divided into the sum of two square integers in 2^{n-1} ways.

2. Shew that

$$(x^2 + xy + y^2)(a^2 + ab + b^2)$$

$$= (ax - by)^2 + (ax - by)(ay + bx + by) + (ay + bx + by)^2$$

and that, therefore, the product of any number of factors of the form $x^2 + xy + y^2$ can be written in the form $X^2 + XY + Y^2$.

3. The sum of p terms of an arithmetical progression is q and the sum of q terms is p; shew that the sum of $p - q$ terms is

$$\left(\frac{2q}{p} + 1\right)(p - q).$$

4. Solve the equations

$$x^2+2yz=247,$$

$$y^2+2zx=235,$$

$$z^2+2xy=247,$$

5. If

$$ax+by=x+y+xy=x^2+y^2-1=0,$$

then

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{(a-b)^2}.$$

XXXII

1. For what values of x is the expression $65x^2-241x+86$ equal to zero? For what values negative and for what values positive? Find also the minimum of the function and represent the function graphically.

2. Shew by mathematical induction that

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots\ldots \text{ to } n \text{ terms}=\frac{1}{2}\left[\frac{1}{1.2}-\frac{1}{(n+1)(n+2)}.\right]$$

3. Simplify

$$\frac{1}{\sqrt[3]{7}-\sqrt[3]{5}},\ \frac{1}{\sqrt[3]{3}+\sqrt[3]{5}+\sqrt[3]{7}}.$$

4. Solve

$$x^2+xy+zx=18,$$

$$y^2+yz+yx=27,$$

$$z^2+zx+zy=36,$$

5. If

$$(a^2-bc)x+(b^2-ca)y+(c^2-ab)z=0$$

and

$$x+y+z=0$$

prove that

$$ax+by+cz=0.$$

XXXIII

1. If a^2+1, a^2-1 measure the hypotenuse and the base of a right-angled triangle, the remaining side is measured by $2a$. Hence find right-angled triangles whose sides are measured by integers.

2. Shew that

$$(a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz)$$
$$=(ax+by+cz)^3+(bx+cy+az)^3+(cx+ay+bz)^3$$
$$-3(ax+by+cz)(bx+cy+az)(cx+ay+bz).$$

3. Employ the theory of the geometrical progression to find the value of $0{\cdot}\dot{2}31\dot{7}$ and $0{\cdot}75\dot{3}8\dot{9}$.

4. Solve

$$x^2-(y-z)^2=a^2,$$
$$y^2-(z-x)^2=b^2,$$
$$z^2-(x-y)^2=c^2.$$

5. If $ax+by+cz=0$ then the value of

$$\frac{ax^2+by^2+cz^2}{bc(y-z)^2+ca(z-x)^2+ab(x-y)^2}$$

is independent of x, y, z.

XXXIV

1. Between what limits must real values of x be taken to make $2+3x-5x^2$ positive?

Find the maximum of the function and construct the graph.

2. Assuming that

$$(1+x)^{n-1}=1+\frac{(n-1)}{1}x+\frac{(n-1)(n-2)}{1.2}x^2+\ldots\ldots$$
$$+\frac{(n-1)(n-2)\ldots\ldots(n-r)}{1.2.3\ldots\ldots r}x^r+\ldots\ldots+x^{n-1},$$

where n is a positive integer, shew that

$$(1+x)^n = 1+\frac{n}{1}\,x+\frac{n(n-1)}{1.2}x^2+\ldots\ldots$$
$$+\frac{n(n-1)\ldots\ldots(n-r+1)}{1.2\ldots\ldots r}x^r+\ldots\ldots+x^n$$

and infer the truth of the binomial theorem for a positive integral exponent.

3. Outline the argument which leads to a meaning for the symbols

$$x^0,\ x^{-3},\ x^{\frac{1}{3}}.$$

4. Solve

$$x^2+(y-z)^2=a^2,$$
$$y^2+(z-x)^2=b^2,$$
$$z^2+(x-y)^2=c^2.$$

5. Given the three equations

$$x=cy+bz,\ y=az+cx,\ z=bx+ay,$$

find the different forms of the values of the ratios $x:y:z$, and shew that

(1) $\dfrac{x^2}{1-a^2}=\dfrac{y^2}{1-b^2}=\dfrac{z^2}{1-c^2}$;

(2) $a^2+b^2+c^2+2abc=1$.

XXXV

1. Shew that

$$(m^2-n^2)^2+(2mn)^2=(m^2+n^2)^2$$

and derive a series of sets of integers which can measure the sides of a right-angled triangle.

2. Shew that the sum, the difference, the product, and the quotient of any two complex numbers is complex.

State similar properties for

(1) positive integers;

(2) integers;

(3) rational real numbers.

3. Find the first factor and the sum of the factors in the nth term of the series

$$1+3.5+7.9.11+13.15.17.19+\ldots\ldots$$

4. Solve

$$\begin{aligned} x^2-yz &= -11, \\ y^2-zx &= 1, \\ z^2-xy &= 13. \end{aligned}$$

5. If

$a(y+z)=x$, $b(z+x)=y$, $c(x+y)=z$, find the different forms of the values of the ratios $x:y:z$ and shew that

(1) $bc+ca+ab+2abc=1$.

(2) $\dfrac{x^2}{a(1-bc)}=\dfrac{y^2}{b(1-ca)}=\dfrac{z^2}{c(1-ab)}$

XXXVI

1. Prove that the least value of the function x^2+px+q is obtained by putting for x one half the sum of the roots of the equation $x^2+px+q=0$.

2. Prove that $a^2+b^2>2ab$ unless $a=b$.

Hence shew that *the arithmetical mean of two positive numbers is greater than their geometrical mean.*

Illustrate both theorems geometrically.

3. Arrange in order of magnitude, without actually finding the approximate roots,

$$2^{\frac{1}{2}},\ 3^{\frac{1}{3}},\ 4^{\frac{1}{4}},\ 5^{\frac{1}{5}},\ 6^{\frac{1}{6}}.$$

4. Find in how many ways the letters of the word *fabric* may be arranged so that (i) the two vowels may not be together; (ii) the two vowels may be separated by two consonants.

5. Find the coefficient of x^{n-r} in the product

$$(x+a_1)(x+a_2)\ldots\ldots(x+a_n)$$

and show what this becomes if $a_1 = a_2 \ldots\ldots = a_n$.

XXXVII

1. Shew that

$$(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+w^2)$$
$$= (ax-by+cz-dw)^2+(ay+bx-cw-dz)^2$$
$$+(az-bw-cx+dy)^2+(aw+bz+cy+dx)^2.$$

Interpret as a theorem in integral numbers.

2. If a, b, c are positive and not all equal then

$$a^2+b^2+c^2 > bc+ca+ab;$$
$$(a+b)(b+c)(c+a) > 8abc;$$
$$a^4+b^4+c^4 > abc(a+b+c);$$
$$(bc+ca+ab)^2 > 3abc(a+b+c).$$

3. Shew that the cube of any integer is the difference of two square integers.

Find the two square integers whose difference is 512.

4. The two solutions of the equations

$$y^2 = 4ax,$$

$$y = mx + n,$$

will be identical if $n = \frac{1}{m}$.

In this case find the double solution.

5. Find the number of terms in the expansion of

$$(a + b + c + d + e)^8$$

and find the coefficient of $a^3bc^2d^2$.

XXXVIII

1. The sides of a rectangle are a and b; from the adjacent sides at two opposite corners a length x is cut off. Find the maximum value of the parallelogram determined by the four points on the sides.

2. The sum of any fraction with positive terms and its reciprocal is greater than 2.

Deduce that, if the numbers involved are all positive

(1) $\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{a}{x} + \frac{b}{y}\right) > 4$

(2) $\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right)\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right) > 9.$

3. Examine, for the number of different values, each of the following:

$$5^{\frac{1}{2}},\ 10^{\frac{1}{3}},\ 7 + 11^{\frac{1}{2}} + 17^{\frac{1}{3}}.$$

4. On the circumference of a circle 11 points are taken; these points are joined in all possible ways. Find the number of chords thus formed and the number of triangles having as sides complete chords.

5. Find the sum of the products of the coefficients taken two together in the expansion of $(1 + x)^n$ where n is a positive integer.

XXXIX

1. Shew that

$$(ay - bx)^2 + (bz - cy)^2 + (cx - az)^2 + (ax + by + cz)^2$$
$$= (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)$$

Interpret as a theorem in integral numbers.

2. The sum of two positive variable numbers is constant; shew that their product is greatest when the numbers are equal. Shew also that the more nearly equal are the numbers the greater is the product. Illustrate geometrically.

3. Find the sum of n terms of the series:

(1) $5.7 + 8.11 + 11.15 + \ldots\ldots$

(2) $5.7.10 + 8.11.15 + 11.15.20 + \ldots\ldots$

4. The two solutions of the equations

$$x^2 + y^2 = r^2$$
$$y = mx + n$$

will be identical if $n^2 = r^2(1 + m^2)$.

5. Shew that

$$1 - n\frac{x+1}{x+n} + \frac{n(n-1)}{1.2}\frac{x^2+2x}{(x+n)^2} - \frac{n(n-1)(n-2)}{1.2.3}\frac{x^3+3x^2}{(x+n)^3} + \ldots\ldots$$

is equal to zero, n being a positive integer.

XL

1. Between what limits must m lie if the polynomial

$$(m-2)x^2 - 2(2m-3)x + (5m-6)$$

is positive for all values of x?

2. Employ the result of Ex. 2, xxxix (p. 155) to find

(1) The maximum of $21 - 4x - x^2$;

(2) The maximum of $3 + x - 2x^2$;

(3) The minimum of $x^2 - 5x + 6$;

(4) The minimum of $2x^2 - 13x + 15$;

(5) The maximum of $x\sqrt{(a^2 - x^2)}$;

indicating any restrictions put upon the values of x

3. Rationalize the equation

$$x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0$$

and compare the significance of the resulting equation with that of the proposed.

4. In how many different ways can 12 different things be divided into 4 groups of 3 things each?

5. Expand to four terms in ascending powers of x

$$\frac{1}{(1-x)^2\sqrt[3]{(1+x)}}, \quad \frac{1+x}{(1-x)^3\sqrt{1+2x}}.$$

XLI

1. Construct the linear function of x which is

(1) Equal to 5 when $x = 1$ and equal to 7 when $x = 2$;

(2) Equal to A when $x = a$ and equal to B when $x = b$.

In the former case represent the function graphically.

2. Find the rectangle of maximum area inscribed in a given circle.

3. Find the sum of the products two together of the first n odd numbers.

4. Find the condition that the two solutions of the equations

$$ax^2 + 2\,hxy + by^2 = c,$$

$$lx + my = p,$$

may be identical.

5. Prove that

$$(1+x)^n = 2^n\left\{1 - h_1 \cdot \frac{1-x}{1+x} + h_2\left(\frac{1-x}{1+x}\right)^2 - \ldots\ldots\right\}.$$

where $h_r = \dfrac{n(n+1)\ldots\ldots(n+r-1)}{1.2.\ldots.r}$.

XLII

1. Find the maximum value of

$$\frac{1}{x^2+2x+2}$$

Represent the function graphically for values of x between -2 and $+5$.

2. Of all triangles with the same perimeter and the same base, find that one which has the greatest area.

3. The men in a regiment can be arranged in a column twice as long as it is wide. If their number were 224 less they could be arranged in a hollow square four deep, having in each outer side of the square as many men as there were in the length of the column. Find the number of men.

4. Find the number of ways in which $3n$ things may be divided into n parcels of three each.

5. Shew that

$$\frac{1+x}{(1-x)^2} = 1 + 3x + 5x^2 + 7x^3 \ldots\ldots \textit{in inf.}$$

first by employing the binomial expansion, and second by summing the series on the right.

XLIII

1. Construct the quadratic function of x which is

(1) Equal to 1 when $x=1$, equal to -1 when $x=2$, equal to -2 when $x=3$;

(2) Equal to l when $x=a$, equal to m when $x=b$, equal to n when $x=c$.

In the former case represent the function graphically.

2. If $x+y$ is constant find when x^2+y^2 is a minimum.

Illustrate geometrically.

3. Find the sum of n terms of the series:

(1) $1+\frac{5}{3}+\frac{9}{9}+\frac{13}{27}+\ldots\ldots$

(2) $\frac{x}{r}+\frac{x+y}{r^2}+\frac{x+2y}{r^3}+\ldots\ldots$

4. If the equations

$$lx+my=1$$

$$ax^2+by^2=1$$

have their two solutions equal then

$$\frac{l^2}{a}+\frac{m^2}{b}=1$$

and the solution is $x=\frac{l}{a}$, $y=\frac{m}{b}$.

5. Shew that

$$\frac{1+x}{(1-x)^3}=1+2^2x+3^2x^2+4^2x^3\ldots\ldots \textit{ in inf.}$$

first by employing the binomial expansion, second by summing the series on the right.

XLIV

1. Find the greatest and least values of

$$\frac{x^2+x+1}{x^2-x+1}$$

2. From a point in the base of an isosceles triangle perpendiculars are drawn to the opposite sides. Find when the sum of the squares of these perpendiculars is a minimum.

3. To complete a certain work A requires m times as many days as B and C together; B requires n times as many as C and A together; and C requires p times as many as A and B together. Compare the times in which each would complete the work, and shew that

$$\frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{p+1}=1.$$

4. In a plane are n points, no four lying on one circle, and through each set of three is described a circle. Find the number of intersections of these circles exclusive of the original points, each circle being supposed to cut every other circle.

5. Find the coefficient of x^n in the expansion of

$$\frac{1+\ x+x^2}{1-2x+x^2}$$

XLV

1. If

$$x^2=px+q$$

shew that it may be assumed that

$$x^n=\mathrm{L}x+\mathrm{M}$$

where L and M are constants.

If α and β are the roots of the given equation find L and M shewing that

$$x^n=\frac{\alpha^n-\beta^n}{\alpha-\beta}x+\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}q.$$

2. A triangle has for base the diameter of a circle and its vertex is on the circle. Find when the sum of the two sides is a maximum.

3. If S be the sum and s^2 the sum of the squares of the terms of a geometrical progression continued to infinity, prove that the sum of the first n terms of the progression is

$$S\left\{1-\left(\frac{S^2-s^2}{S^2+s^2}\right)^n\right\}.$$

4. Find the condition that the roots of the equation $ax^2+2bx+c=0$ may be the roots of the equation $a'x^2+2b'x+c'=0$ increased by the same quantity.

5. Shew that the coefficient of x^2 in the expansion of

$$\frac{1+x}{1+x+x^2}$$

is 0 if r is greater by 1 than a multiple of 3.

XLVI

1. Find the maximum and minimum values of

$$\frac{x^2-x-8}{x^2+x-2}, \quad \frac{x^2+4x-4}{x^2+2x-2}.$$

2. Shew that, if $a+b+c$ is constant, $bc+ca+ab$ has its greatest value and $a^2+b^2+c^2$ its least value when $a=b=c$.

3. A merchant employs a false balance both in buying and in selling and thereby gains at the rate of 11 per cent. more on his outlay than if his balance were true. If, however, with the same scales he had, in buying and in selling, weighed to his disadvantage, he would have neither gained nor lost. Find his legitimate gain per cent.

4. Find the number of ways in which ten persons may distribute themselves among four rooms, with at least one person in each room.

5. Find the sum to infinity of the series,

$$1+\frac{2}{6}+\frac{2.5}{6.12}+\frac{2.5.8}{6.12.18}+\cdots\cdots$$

XLVII

1. Assuming that for all integral values

$$1^2+2^2+3^2+\ldots\ldots+n^2=An^3+Bn^2+Cn+D$$

where A, B, C, D are constants, *i.e.*, do not involve n, find A, B, C, D.

2. A point is taken within an equilateral triangle. Find the position of the point if the sum of the squares of the perpendiculars from it to the three sides is a minimum.

3. Between a and b are inserted three arithmetical and three geometrical means. Shew that each arithmetical mean is greater than the corresponding geometrical mean.

4. Find the condition that the value of the fraction

$$\frac{ax^2+bx+c}{a'x^2+b'x+c'}$$

may be independent of the value of x.

5. Shew that

$$\sqrt[3]{\frac{6}{7}}=1-\frac{1}{3}\cdot\frac{1}{6}+\frac{1.4}{3.6}\cdot\left(\frac{1}{6}\right)^2-\frac{1.4.7}{3.6.9}\cdot\left(\frac{1}{6}\right)^3+\ldots\ldots$$

XLVIII

1. Shew that if a, b, c are positive numbers, the value of $a^3+b^3+c^3$ exceeds that of $3abc$ unless $a=b=c$.

Cor. If a, b, c are positive and not all equal

$$\frac{a+b+c}{3}>(abc)^{\frac{1}{3}},$$

or *the arithmetical mean of three positive numbers, not all equal, is greater than their geometrical mean.*

2. Two straight lines OX, OY are at right angles to each other, and through a point P, the distances of which from these lines are 3 and 4 inches, is drawn a straight line APB meeting OX, OY at A, B. Find the minimum area of the triangle OAB.

3. Find two numbers such that their sum multiplied by the sum of their squares shall be 272, while their difference multiplied by the difference of their squares shall be 32.

4. In how many ways may 18 different things be put up in 3 packages of 2 each and 4 packages of 3 each?

5. Find the value of

$$2+\frac{5}{3.2\,!}+\frac{5.7}{3^2.3\,!}+\ldots\ldots \textit{in inf.}$$

XLIX

1. Assuming that for all values of n

$1.2.3+3.4.5+\ldots\ldots+(2n-1)2n(2n+1)=An^4+Bn^3+Cn^2+Dn+E$ find the expression for the sum of n terms of the series.

2. Prove that

$$(a+b+c)(a^2+b^2+c^2)>9abc;$$

$$\left(\frac{l}{a}+\frac{m}{b}+\frac{n}{c}\right)\left(\frac{a}{l}+\frac{b}{m}+\frac{n}{c}\right)>9.$$

3. An A.P., a G.P., and an H.P., in each of which the first two terms are a, b, have for third terms x, y, z respectively; shew that

$$y(x-3z)^2+4z(y^2+zx)=12yz^2.$$

4. Being given that $ax^2+2bx+c$ vanishes for three (or more) given values of x shew that $a=0$, $b=0$, $c=0$.

If $ax^2+2bx+c$ is equal to $a'x^2+2b'x+c'$ for more than two values of x, shew that these quantities are equal for all values of x.

5. Find the sum of the first m coefficients in the expansion of each of the following.

$$(1-x)^{\frac{1}{2}},\ (1-x)^{-\frac{3}{2}},\ (1-x)^{-\frac{2}{3}}.$$

L

1. ABCD is a rectangle, APQ meets BC in P and DC produced in Q. Find the position of APQ when the sum of the areas ABP, PCQ is a minimum.

2. From the corollary in Ex. 1, xlviii (p. 161) deduce the theorems: *The product of three positive numbers whose sum is constant is greatest when the three numbers are equal, and if the product of three positive numbers is constant their sum is least when the three numbers are equal.*

3. If

$$\frac{a}{x}(b-c)+\frac{b}{y}(c-a)+\frac{c}{z}(a-b)=0,$$

then

$$\frac{x}{a}(y-z)+\frac{y}{b}(z-x)+\frac{z}{c}(x-y)=0.$$

4. In how many ways can $p+2n$ different things be divided into three parcels containing p, n, n things respectively?

5. If a_r is the coefficient of x^r in the expansion of $(1+x)^n$, then shall

$$a_0-a_1+a_2-a_3+\ldots\ldots+(-1)^r a_r=(-1)^r\frac{r+1}{n}\cdot a_{r+1}.$$

LI

1. Find the conditions that

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$

may be the square of a rational function of x, y, z.

2. The total length of the edges of a box of the ordinary form is to be 24 feet. Find the length of each edge if the volume is to be the greatest possible.

3. If a, b, c are in A.P.; α, β, γ in H.P.; $a\alpha$, $b\beta$, $c\gamma$ in G.P.; then

$$\frac{\alpha}{\gamma}+\frac{\gamma}{\alpha}=\frac{a}{c}+\frac{c}{a}.$$

4. Eliminate x and y from the equations

$$x+\frac{1}{x}=a,\quad y+\frac{1}{y}=b,\quad xy+\frac{1}{xy}=c.$$

5. Between what two positive integers does the value of $(\sqrt{29}+5)^{2n}$ lie?

LII

1. If p is greater than 1, then for real values of x the expression

$$\frac{x^2-2x+p^2}{x^2+2x+p^2}$$

lies between $$\frac{p-1}{p+1} \text{ and } \frac{p+1}{p-1}.$$

2. Of all rectangular parallelopipeds with the same volume find that of minimum surface.

3. If

$$x\left(1-\frac{myz}{x^3}\right) = y\left(1-\frac{mzx}{y^3}\right) = z\left(1-\frac{mxy}{z^3}\right),$$

and x, y, z are unequal, then each member of these equations is equal to

$$x+y+z-m.$$

4. Upon a plane are drawn n lines, of which p pass through one and the same point. Shew that the total number of determined points is

$$\tfrac{1}{2}(n-p)(n+p-1)+1.$$

5. In the expansion of $\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$ shew that the coefficients of x^{2r} and x^{2r+1} are the same.

LIII

1. If

$$a - \frac{yz}{x} = b - \frac{zx}{y} = c - \frac{xy}{z},$$

where no one of x, y, z is zero, then

$$a + \frac{y^2 - z^2}{b - c} = b + \frac{z^2 - x^2}{c - a} = c + \frac{x^2 - y^2}{a - b}.$$

2. Find the rectangular parallelopiped of maximum volume inscribed in a sphere.

3. If the nth terms of two arithmetical progressions are $a - bn$ and $b + an$, shew that they have a common sum for the same number of terms if $\frac{4b}{a+b}$ is a negative integer.

4. Eliminate x and y from the equations

$$x^2 + xy = l^2, \quad y^2 + yx = m^2, \quad x^2 + y^2 = n^2.$$

5. Find the coefficient of x^n in the expansion of $\frac{1+x}{(1-x)^3}$, and employ the result to find the sum of the squares of the first n natural numbers.

LIV

1. If

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}$$

and n is an odd positive integer, then

$$\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} = \frac{1}{a^n + b^n + c^n}.$$

2. Shew that of all triangles with a given perimeter the equilateral triangle is of maximum area.

3. If

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0,\ \frac{a}{x^2}+\frac{b}{y^2}+\frac{c}{z^2}=0.$$

then

$$(b+c)x+(c+a)y+(a+b)z=0.$$

4. If m like cards are to be placed in $n+1$ numbered envelopes and no restriction is made as to the number to be placed in any envelope, find the number of ways in which the distribution can be made.

5. The coefficient of x^{n+r-1}, where n, r are positive integers, in the expansion of

$$\frac{(1+x)^n}{(1-x)^3}$$

is $2^{n-3}\left\{n+(n+2r)(n+2r+2)\right\}$.

LV

1. If

$$\alpha^3+p\alpha+q=0,$$

$$\beta^3+p\beta+q=0,$$

$$\gamma^3+p\gamma+q=0,$$

then

$$\alpha+\beta+\gamma=0.$$

2. Find the minimum values of

$$(x-2)^2(11-2x),\ (x-3)^2(7-x).$$

3. If P is the continued product of n quantities in G.P., S their sum and T the sum of their reciprocals, then

$$P^2=\left(\frac{S}{T}\right)^n.$$

4. Eliminate x, y, z, from the equations

$$x(y+z)=a^2,\quad y(z+x)=b^2,\quad z(x+y)=c^2,\quad xyz=d^3.$$

5. If p, q, s denote respectively the product, the quotient and the sum of two numbers then

$$p = s^2(q - 2q^2 + 3q^3 - \ldots\ldots \textit{ in inf.}).$$

LVI

1. If

$$\frac{a-b}{1+ab} + \frac{c-d}{1+cd} = 0$$

prove that

$$\frac{b-c}{1+bc} + \frac{d-a}{1+ad} = 0.$$

Give a trigonometrical interpretation.

2. Of all cylinders inscribed in a sphere find that of maximum volume.

3. If

$$y^3 - z^3 = ayz, \quad z^3 - x^3 = bzx, \quad x^3 - y^3 = cxy,$$

then

$$\frac{a^3 + b^3 + c^3}{abc} = \frac{x^3 + y^3 + z^3}{xyz}.$$

4. In each of three bags are ten tickets numbered from 1 to 10. One ticket is drawn from each bag and the sum of the numbers drawn is 21. In how many ways might this have happened?

5. Find the remainder after n terms of the expansion of $(1-x)^{-1}$, $(1-x)^{-2}$, $(1-x)^{-3}$.

LVII

1. Shew that $ax^2 + 2hxy + by^2$ and $hx^2 - 2(a-b)xy - hy^2$ cannot have a common factor unless the first of these expressions is a square.

2. Find the maximum volume of a circular cone inscribed in a sphere.

3. If

$$\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz}$$

and p, q, r are in arithmetical progression then x, y, z are in harmonical progression.

4. Eliminate x, y, z from the equations

$$\frac{y}{x}+\frac{x}{y}=a, \quad \frac{z}{y}+\frac{y}{z}=b, \quad \frac{x}{z}+\frac{z}{x}=c.$$

5. Find the value of

$$(n+1)(n+2)(n+3).1.2+n(n+1)(n+2).2.3$$
$$+\ldots\ldots+1.2.3.(n+1)(n+2).$$

LVIII

1. If $x+y+z=xyz$ then

$$\frac{y+z}{1-yz}+\frac{z+x}{1-zx}+\frac{x+y}{1-xy}=\frac{(y+z)(z+x)(x+y)}{(1-yz)(1-zx)(1-xy)}.$$

Interpret trigonometrically.

2. From Ex. 2, xlvi (p. 160) and the identity

$$(a+b+c)^3=3(a+b+c)(a^2+b^2+c^2)-2(a^3+b^3+c^3)+6abc,$$

infer the result of Ex. 1, xlviii (p. 161).

3. If

$$a^2-\alpha^2=b^2-\beta^2=c^2-\gamma^2,$$

prove that

$$\frac{b\gamma-c\beta}{a-\alpha}+\frac{c\alpha-a\gamma}{b-\beta}+\frac{a\beta-b\alpha}{c-\gamma}=0.$$

4. A candidate writes on three papers, each carrying a total of m marks. In how many ways may he make $2m$ marks?

5. Find the greatest term in the expansion of $(1+x)^{\frac{9}{2}}$ when $x=\frac{5}{6}$.

LIX

1. If g_1, g_2, g_3 are the G.C.M.'s, l_1, l_2, l_3 the L.C.M.'s of b and c, c and a, a and b respectively, G the G.C.M. and L the L.C.M. of the three a, b, c, shew that

$$L = \frac{abc\,G}{g_1\,g_2\,g_3}; \quad \frac{L}{G} = \sqrt{\left(\frac{l_1\,l_2\,l_3}{g_1\,g_2\,g_3}\right)}$$

2. A cylinder is inscribed in a right circular cone. Find when the cylinder is of maximum volume.

3. If the sum of m terms of an A.P. be equal to the sum of the next n terms and also equal to the sum of the next p terms, shew that

$$\frac{(m+n)\,(p-n)}{np} = \frac{(n+p)\,(n-m)}{mn}.$$

4. Eliminate x, y, z from the equations

$$y^2 + z^2 = ayz$$

$$z^2 + x^2 = bzx$$

$$x^2 + y^2 = cxy$$

if no one of x, y, z is zero.

5. If $f(r)$ is the coefficient of x^r in

$$1 + nx(1+x) + \frac{n(n-1)}{1.2}x^2(1+x)^2 + \ldots.$$

n being a positive integer, then

$$2\{f(0) + f(1) + \ldots\ldots + f(n-1)\} + f(n) = 3^n.$$

LX

1. If $xy+yz+zx=1$, then

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

Interpret trigonometrically.

2. A cylinder is inscribed in a cone. Find when its curved surface is a maximum.

3. Having given

$$(y-b)^2=a(x-a)$$

$$(x-a)^2=b(y-b)$$

$$x^2+y^2=c^2$$

shew that

$$a^{\frac{2}{3}}+b^{\frac{2}{3}}=c^{\frac{2}{3}}.$$

4. On each side of a square 5 points are taken and these points are joined in all possible ways; prove that, supposing no three of the joining lines to meet at a point, the square is divided into 4376 parts.

5. If

$$\frac{1}{(x-a)(x-b)}$$

is expanded in a series of ascending powers of x, shew that the coefficient of x^{n+1} is

$$\frac{a^n-b^n}{a-b}\cdot\frac{1}{a^nb^n}.$$

ADDITIONAL EXAMPLES

FOR

CANDIDATES FOR SCHOLARSHIPS

AT THE

UNIVERSITY EXAMINATIONS

I

1. Solve

$$\frac{48}{x^2 - 2x - 3} - \frac{36}{x^2 - x - 2} = x^2 - 3x + 2.$$

2. If $a + b + c = 0$ shew that

$$\frac{(a^2 + b^2 + c^2)(a^5 + b^5 + c^5)}{(a^3 + b^3 + c^3)(a^4 + b^4 + c^4)} = \frac{5}{3}.$$

3. If

$$x^4 + y^4 + z^4 + y^2z^2 + z^2x^2 + x^2y^2 = 2xyz(x + y + z)$$

and xyz are all real, then $x = y = z$.

4. Find what relation must exist among the coefficients of

$$ax^2 + 2bxy + by^2 + 2gx + 2fy + c$$

in order that this expression may be the product of two factors linear in x, y.

Factor $x^2 + y^2 + z^2 - yz - zx - xy$.

5. Find the number of ways of putting m things in $n + 1$ places, (i) supposing the things alike, (ii) supposing no two of the things to be alike.

II

1. Solve

$$7 \cdot \frac{x^2+1}{x+1} = 5 \cdot \frac{xy+1}{y+1},$$

$$7 \cdot \frac{y^2+1}{y+1} = 10 \cdot \frac{xy+1}{x+1}.$$

2. The equations

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1,$$

$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z}=0,$$

are equivalent to only two independent equations if $a+b+c=0$.

3. Examine the variation of

$$\frac{x^2-5x+6}{x^2-8x+16}$$

as x passes through all real values, and construct the graph of this function.

4. If $x+y+z$ is a factor of

$$ax^3+by^3+cz^3+dyz^2+d'zy^2+ezx^2+e'xz^2+fxy^2+f'yx^2+gxyz,$$

then

$$b-c+d-d'=c-a+e-e'=a-b+f-f'=0,$$

and

$$a+b+c=d'+e'+f'-g.$$

5. If ${}_nC_r$ denotes the number of combinations of n things r at a time shew that

$${}_mC_r+{}_nC_1\cdot{}_mC_{r-1}+{}_nC_2\cdot{}_mC_{r-2}+\ldots\ldots+{}_nC_r={}_{m+n}C_r$$

(i) from the theory of combinations, (ii) from binomial expansions.

III

1. Solve

$$x+y+z=1,$$
$$ax+by+cz=-(a+b+c),$$
$$a^2x+b^2y+c^2z=(a+b+c)^2,$$

and shew that

$$a^4x+b^4y+c^4z=(a^2x+b^2y+c^2z)^2.$$

2. If

$$\frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ca}+\frac{a^2+b^2-c^2}{2ab}=1,$$

then shall

$$(b+c-a)(c+a-b)(a+b-c)=0,$$

and, of the three fractions appearing in the right member of the given equation, two must be equal each to $+1$ and the third to -1.

Give a trigonometrical interpretation.

3. Shew by mathematical induction that

$$2.7^n+3.5^n-5$$

is divisible by 24.

4. The sums of the first n_1, n_2, n_3 terms of the same arithmetical series are S_1, S_2, S_3, prove that

$$\frac{S_1}{n_1}(n_2-n_3)+\frac{S_2}{n_2}(n_3-n_1)+\frac{S_3}{n_3}(n_1-n_2)=0.$$

5. Find the number of ways in which m black balls and n white balls may be placed in line so that there may be $2r-1$ contacts of a black ball with a white ball.

IV

1. Solve.

$$x^3+y^3+z^3=3xyz, \quad x-a=y-b=z-c.$$

2. If $x+y+z=xyz$ prove that

$$\left(\frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{z}{x}+\frac{x}{z}+2\right)^2=(1+x^2)(1+y^2)(1+z^2).$$

3. Find the maximum and minimum values of

$$\frac{x^2+4x-36}{2(x-5)}.$$

4. By means of the ordinary process of finding the H.C.F., shew that $x^2+11x+33$ and $x^3+9x^2+12x-61$ are without common factor, and shew that the operation makes it possible to find two polynomials which multiplied into the given polynomials will give polynomials which differ by a quantity not involving x.

Result: $(x^2+11x+33)(x^2+4x-1)-(x^3+9x^2+12x-61)(x+6)=3.$

5. Find the value, to m terms, of the series

$$m-(m-1)n+(m-2)\frac{n(n-1)}{1.2}-(m-3)\frac{n(n-1)(n-2)}{1.2.3}+\ldots\ldots$$

V

1. The solution of a certain problem is made to depend upon that of the simple equation

$$ax-b=0.$$

In this equation the coefficient a is a number which undergoes change. Find what the root becomes as x approaches the value zero.

2. Shew that the equations

$$x+y+z=a,$$
$$x^2+y^2+z^2=b^2,$$
$$z^3+y^3+z^3-3xyz=c^3,$$

are not independent but give a relation among a, b, c.

3. Find the cone of minimum volume, circumscribed to a given sphere.

4. Construct the cubic function of x which is equal to l when $x=a$, equal to m when $x=b$, equal to n when $x=c$, and equal to r when $x=d$.

5. Find the value of

$$1.\ n(n+1)+2.(n-1)n+3(n-2)(n-1)+\ldots\ldots+n.1.2.$$

VI

1. The solution of a certain problem is found to depend upon that of the equation

$$ax^2 + 2bx + c = 0.$$

In this equation the coefficient a is a number which undergoes change. Find what the roots become as a approaches the value zero.

2. Shew, without actual elimination, that the result of eliminating y from the equations

$$ax^2 + 2hxy + by^2 = c,$$

$$a'x^2 + 2h'xy + b'y^2 = c',$$

is an equation of degree four in x in which occur only even powers of x.

3. (a) From the inequality (Ex. 2, xxxvii, p. 153)

$$(bc + ca + ab)^2 > 3abc(a + b + c)$$

and the result of Ex. 2, xlvi (p. 160), shew that if $a + b + c$ is constant the value of abc is greatest when $a = b = c$.

(b). Deduce the theorem stated in (a) from the identity

$$2(a+b+c)^3 - 7\left\{a(b-c)^2 + b(c-a)^2 + c(a-b)^2\right\}$$
$$- \left\{(b+c)(b-c)^2 + (c+a)(c-a)^2 + (a+b)(a-b)^2\right\} = 54abc.$$

4. A rope 512 ft. long has one end fastened to a corner of a house 25 ft. square, around which it is then completely wrapped. How far will a man, holding the other end of the rope and keeping it stretched, walk before the rope is re-wrapped around the house in the contrary direction?

5. There are $p + q$ numbers $\alpha, \beta, \gamma, \ldots\ldots$ of which p are even and q odd. Shew that the sum of the products, taken three together, of the quantities

$$(-1)^\alpha, (-1)^\beta, (-1)^\gamma, \ldots\ldots$$

is

$$-\tfrac{1}{6}\left\{(q-p)^3 - 3(q^2 - p^2) + 2(q-p)\right\}.$$

VII

1. Solve

$$(x+y)(xy+1)=18xy,$$

$$(x^2+y^2)(x^2y^2+1)=208x^2y^2.$$

2. If

$$\frac{a-ny+mz}{l'} = \frac{b-lz+nx}{m'} = \frac{c-mx+ly}{n'}$$

then

$$\frac{x-\dfrac{m'c-n'b}{ll'+mm'+nn'}}{l} = \frac{y-\dfrac{n'a-l'c}{ll'+mm'+nn'}}{m} = \frac{z-\dfrac{l'b-m'a}{ll'+mm'+nn'}}{n}.$$

3. If the values of x, y, x', y' are all real and

$$1+xx'+yy'=\sqrt{(1+x^2+y^2)}\sqrt{(1+x'^2+y'^2)},$$

then

$$x=x',\ y=y'.$$

4. Shew that there is an infinite number of solutions of the equations,

$$\frac{lx+my}{nz} = \frac{nz+lx}{my} = \frac{my+nz}{lx} = x+y+z,$$

and explain this result.

5. If n is a positive integer find the value of

$$(1)\quad 1.2.n+2.3.\frac{n(n-1)}{2!} + 3.4.\frac{n(n-1)(n-2)}{3!} + \ldots\ldots$$

$$(2)\quad 1.2.n+3.4.\frac{n(n-1)}{2!} + 5.6.\frac{n(n-1)(n-2)}{3!} + \ldots\ldots$$

VIII

1. Solve

$$a^2(a^2-x^2) = b^2(b^2-y^2) = c^2(c^2-z^2)$$

$$= bcyz+cazx+abxy.$$

2. If

$$yz+zx+xy=1,$$

then

$$x\left\{\frac{(1+y^2)(1+z^2)}{1+x^2}\right\}^{\frac{1}{2}}+y\left\{\frac{(1+z^2)(1+x^2)}{1+y^2}\right\}^{\frac{1}{2}}+z\left\{\frac{(1+x^2)(1+y^2)}{1+z^2}\right\}^{\frac{1}{2}}=2.$$

Give a trigonometrical interpretation.

3. If

$$l^2+m^2+n^2=1 \quad , l'^2+m'^2+n'^2=1,$$

then

$$ll'+mm'+nn'<1.$$

4. If

$$\frac{u}{a+x}+\frac{v}{b+x}+\frac{w}{c+x}+\frac{uvw}{(a+x)(b+x)(c+x)}=0$$

for all values of x, find the values of u, v, w.

5. If x be small compared with N^2, prove that approximately

$$\sqrt{(N^2+x)}=N+\frac{x}{4N}+\frac{Nx}{2(2N^2+x)}.$$

IX

1. Solve

$$\frac{1}{x}\left(\frac{3}{z}-\frac{4}{y}+\frac{7}{yz}\right)+\frac{2}{yz}$$

$$=\frac{1}{y}\left(\frac{3}{x}-\frac{4}{z}-\frac{19}{zx}\right)+\frac{2}{zx}$$

$$=\frac{1}{z}\left(\frac{3}{y}-\frac{4}{x}-\frac{3}{xy}\right)+\frac{2}{xy}=0.$$

2. If

$$a^2x^2+b^2y^2+c^2z^2=0,\ a^2x^3+b^2y^3+c^2z^3=0,$$

$$\frac{1}{x}-a^2=\frac{1}{y}-b^2=\frac{1}{z}-c^2,$$

then

$$a^4x^3+b^4y^3+c^4z^3=0,$$

$$a^6x^3+b^6y^3+c^6z^3=a^4x^2+b^4y^2+c^4z^2.$$

3. Shew that the formula

$$9(a^3+b^3+c^3) - (a+b+c)^3$$

$$=(4b+4c-a)(b-c)^2+(4c+4a-b)(c-a)^2+(4a+4b-c)(a-b)^2$$

gives the theorem: *If the sum of three positive numbers is constant the sum of their cubes is a minimum when $a=b=c$.*

4. Sum to $2m$ terms and to $2m+1$ terms the series,

$$1^2-3^2+5^2-7^2+\ldots\ldots$$

Write down the expression which will give the sum of n terms whatever positive integer n may be.

5. A candidate writes on three papers, each carrying a total of $6m$ marks. To pass he has to make at least $2m$ marks on each paper and at least $9m$ marks in all. In how many ways may he make the marks necessary to pass?

X

1. (*a*) Shew that a finite solution of two simultaneous linear equations in two unknowns may be either *unique, indeterminate* or *impossible.*

(*b*) Solve the equations.

$$(x-a)(y-b)=(x-a')(y-b')=(x-a'')(y-b'')$$

and shew that the solution is, in general, impossible if

$$ab'+a'b''+a''b=a'b+a''b'+ab'',$$

examining in particular the cases when it becomes indeterminate.

2. Eliminate x, y, z from the equations

$$\frac{y}{z}-\frac{z}{y}=a, \quad \frac{z}{x}-\frac{x}{z}=b, \quad \frac{x}{y}-\frac{y}{x}=c.$$

3. Find the maximum values of

$$\text{(i)}\ (x-1)^2(7-2x) \quad \text{(ii)}\ (x-2)^2(5-x)$$

the variation of x being limited by the condition that the factors are to be positive.

Draw a graph of each of the functions for the values of x in question.

4. The remainder when a polynomial $f(x)$ is divided by $(x-a)(x-b)$ is

$$\frac{f(a)}{a-b}(x-b)+\frac{f(b)}{b-a}(x-a).$$

5. Prove that if p be equal to N^4+x where x is small, then $p^{\frac{1}{4}}$ is approximately equal to

$$\frac{51}{56}N+\frac{5}{56}\frac{p}{N^3}+\frac{27}{14}\frac{Nx}{7p+5N^4}.$$

XI

1. If $m+in$ is a root of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n-0$$

where the coefficients are all real, so also is $m-in$ a root.

2. If

$$(y+z)^2=4a^2yz, \ (z+x)^2=4b^2zx, \ (x+y)^2=4c^2xy,$$

find the relation that must exist among a, b, c.

3. Shew by mathematical induction that if n is a positive integer

$$(1)\ 1-\frac{1}{2}+\frac{1}{3}-\ldots\ldots-\frac{1}{n}\ (n \text{ even}) = 2\left(\frac{1}{n+2}+\frac{1}{n+4}+\ldots\ldots+\frac{1}{2n}\right);$$

$$(2)\ 1-\frac{1}{2}+\frac{1}{3}-\ldots\ldots+\frac{1}{n}\ (n \text{ odd}) = 2\left(\frac{1}{n+1}+\frac{1}{n+3}+\ldots\ldots+\frac{1}{2n}\right).$$

4. The remainder when the polynomial $f(x)$ is divided by $(x-a)(x-b)(x-c)$ is

$$\frac{f(a)}{(a-b)(a-c)}(x-b)(x-c)+\frac{f(b)}{(b-c)(b-a)}(x-c)(x-a)$$
$$+\frac{f(c)}{(c-a)(c-b)}(x-a)(x-b).$$

5. Shew that

$$2^n+\frac{n(n-1)}{1!\,1!}2^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2!\,2!}+\ldots\ldots=\frac{(2n)}{n!\,n!}.$$

XII

1. Solve

$$(x+y+z)^2+yz+zx+xy = a^2-(y+z)^2$$
$$= b^2-(z+x)^2 = c^2-(x+y)^2.$$

2. Prove that if

$$a+b+c+d=0,$$
$$x+y+z+u=0,$$
$$ax+by+cz+du=0,$$

then

$$2(a^4x+b^4y+c^4z+d^4u)=(a^2x+b^2y+c^2z+d^2u)(a^2+b^2+c^2+d^2).$$

3. Find the maximum value of

$$(x-1)(x-2)(7-x)$$

the variation of x being limited by the condition that each factor is to be positive.

Make a graph of the function.

4. If

$$\frac{a+(a+y)x+(a+2y)x^2+\ldots\ldots \textit{in inf,}}{a+(a-y)x+(a-2y)x^2+\ldots\ldots \textit{in inf,}}=b,$$

and if x receive values H.P., shew that the corresponding values of y are in A.P.

5. In a plane lie n straight lines, no two parallel and no three passing through a point. The number of groups of n of their points of intersection, no three in a straight line, is $(n-1)! \div 2$.

XIII

1. Solve

$$(bzx + cxy - ayz)yz = (cxy + ayz - bzx)zx$$
$$= (ayz + bzx - cxy)xy = (xyz)^{\frac{3}{2}}$$

2. If $bz + cy = a,\quad cx + az = b,\quad ay + bx = c,\quad$ then

$$\left(\frac{1-y}{1+y}\cdot\frac{1-z}{1+z}\right)^{\frac{1}{2}} + \left(\frac{1-z}{1+z}\cdot\frac{1-x}{1+x}\right)^{\frac{1}{2}} + \left(\frac{1-x}{1+x}\cdot\frac{1-y}{1+y}\right)^{\frac{1}{2}} = 1\,;$$

$$\frac{1+x}{s}\cdot\frac{1+y}{s}\cdot\frac{1+z}{s} = \frac{1-x}{s-a}\cdot\frac{1-y}{s-b}\cdot\frac{1-z}{s-c}\,.$$

when $2s = a + b + c$.

Interpret trigonometrically.

3. In the sides BC, CA, AB of an equilateral triangle are taken the points P, Q, R such that BP, CQ, AR are equal. Find when the area of the triangle PQR is a minimum.

4. A, B, C, D are four points on a straight line and a ratio of segments AC.BD : BC.AD is formed; how many such ratios may be formed by different choices of segments, and how many distinct values will these ratios present?

5. Shew that

$$2^n - (n-1)2^{n-2} + \frac{(n-2)(n-3)}{2!}2^{n-4} - \frac{(n-3)(n-4)(n-5)}{3!}2^{n-6} + \ldots$$
$$= n + 1.$$

XIV

1. Solve

$$yz = a(y+z)+l,$$

$$zx = a(z+x)+m,$$

$$xy = a(x+y)+n.$$

2. If

$$\frac{1}{1+x+zx}+\frac{y}{1+y+xy}+\frac{yz}{1+z+yz}=1,$$

$$\frac{x}{1+x+zx}+\frac{xy}{1+y+xy}+\frac{1}{1+z+yz}=1,$$

none of the denominators being zero, then $x=y=z$.

3. The equation

$$a^2b^4(x-x')^2+b^2a^4(y-y')^2+(b^2x^2+a^2y^2-a^2b^2)(b^2x'^2+a^2y'^2-a^2b^2)=0$$

in which the unknowns are x, y, is equivalent to the two equations

$$a^2b^2-a^2yy'-b^2xx'=x0, \quad xy'-x'y=0.$$

4. Express as a sum of partial fractions

$$\frac{n\,!}{x(x+1)\ldots\ldots(x+n)}.$$

5. Find the sum of the homogeneous products of n dimensions of the three quantities a, b, c.

XV

1. Assuming the theorem that every polynomial in x will vanish for at least one value of x, real or imaginary, shew that the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n=0$$

has n and only n roots.

Denoting these roots by $a_1, a_2, \ldots\ldots a_n$, shew that

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n=(x-a_1)(x-a_2)\ldots\ldots(x-a_n).$$

2. Eliminate x, y, z from the equations

$$x^2(y+z)=a^3,\ \ y^2(z+x)=b^3,\ \ z^2(x+y)=c^3,$$

$$abc=(y+z)(z+x)(x+y).$$

3. Of all cylinders with a given total surface $2\pi a^2$, find that of maximum volume.

4. Find the sum of all fractions of the form $\dfrac{a^p}{b^q}$ where a and b are given quantities and to p and q are assigned all positive integral values, zero included, which are less than a given integer n.

5. All the diagonals of a convex n-sided polygon are drawn and produced indefinitely. Find the number of internal and the number of external intersections of these diagonals.

XVI

1. If a polynomial $a_0x^n+a_1x^{n-1}+\ldots\ldots+a_n$, of degree n in x vanishes for more than n distinct values of x, then the polynomial vanishes for all values of x, which implies that each coefficient is zero identically.

2. Eliminate x, y, z from the equations

$$p=x-\frac{yz}{x},\ \ q=y-\frac{zx}{y},\ \ r=z-\frac{xy}{z},$$

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0.$$

3. Through a given point a straight line is drawn cutting a given circle in two points. Find when the triangle with the chord as base and centre as vertex is of maximum value.

4. A number which is an exact square is expressed by n digits. Find a formula for the number of digits in its square root.

5. Shew that

$$1+\left(\frac{n}{1}\right)^2\cdot\frac{1}{3}+\left(\frac{n(n-1)}{1.2}\right)^2\cdot\frac{1}{3^2}+\left(\frac{n(n-1)(n-2)}{1.2.3}\right)^2\frac{1}{3^3}+\ldots\ldots$$

$$=\left(\frac{4}{3}\right)^n\left[1+\frac{n(n-1)}{1.1}\cdot\frac{3}{4^2}+\frac{n(n-1)(n-2)(n-3)}{1.2.1.2}\cdot\frac{3^2}{4^4}+\ldots\ldots\right].$$

XVII

1. Solve

$$x^2-yz=a^2,$$

$$y^2-zx=b^2,$$

$$z^2-xy=c^2.$$

2. Shew that if

$$yz+zx+xy=a^2,$$

then

$$\frac{1}{yz(a^2+x^2)}+\frac{1}{zx(a^2+y^2)}+\frac{1}{xy(a^2+z^2)}$$

$$=\frac{2a^2}{xyz\sqrt{(a^2+x^2)(a^2+y^2)(a^2+z^2)}}.$$

3. On the side CD, produced, if necessary, of a square ABCD, it is required to find the point P for which the ratio PA:PB has (i) the greatest value possible, (ii) the least value possible.

4. In extracting the square root of a polynomial in x, shew that, when r terms of the root have been obtained, r additional terms may be found by mere division.

Shew also that, in extracting a numerical square root, when $r+1$ figures of the root have been found, r additional figures may be found by mere division.

5. If $c_0, c_1, c_2, \ldots\ldots c_n$, are the coefficients in the expression of $(1+x)^n$, where n is a positive integer, shew that

$$\frac{c_1}{1}-\frac{c_2}{2}+\frac{c_3}{3}-\ldots\ldots+(-1)^{n-1}\frac{c_n}{n}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}.$$

XVIII

1. Solve

$$\frac{x+y}{1+xy}=\frac{7}{8},$$

$$\frac{x^4+y^4}{1+x^4y^4}=\frac{337}{1312}.$$

2. If $x^2+y^2+z^2+2xyz=1$, prove that

(1) $\left\{(1-y^2)(1-z^2)\right\}^{\frac{1}{2}}+\ldots\ldots+\ldots\ldots=y(1+z)+\ldots\ldots+\ldots\ldots$

(2) $\left\{\frac{(1-x)}{1+x}\cdot\frac{1-y}{1+y}\right\}^{\frac{1}{2}}+\ldots\ldots+\ldots\ldots=1.$

Give a trigonometrical interpretation.

3. (*a*) From the corners of a square piece of cardboard, of side 12 inches (or a inches), equal square pieces are cut, and then the rectangles at the sides are turned upwards about their attached sides so as to form a box open at the top. Find the side of the square pieces detached if the volume of the box is the greatest possible.

(*b*) From the corners of a rectangular piece of cardboard, the sides of which measure $2a$ and $3b$, four equal squares are cut and the rectangles left along the sides are turned upwards about their attached side to form a box. Find the maximum volume of the box.

4. If z, x, y are in G.P. when l is subtracted from each; and x, y, z are in G.P. when m is subtracted from each; and y, z, x are in G.P. when n is subtracted from each, then

$$\frac{1}{l-x}+\frac{1}{m-y}+\frac{1}{n-z}=0.$$

5. Into how many regions is the infinite plane divided by n lines lying in it?

XIX

1. Solve

$$y^2+z^2 - x(y+z)=a^2,$$

$$z^2+x^2 - y(z+x)=b^2,$$

$$x^2+y^2 - z(x+y)=c^2,$$

2. From the equations

$$x^2+x_1^2+x_2^2=mb^2,\quad y^2+y_1^2+y_2^2=ma^2,$$

$$x+x_1+x_2=0,\qquad y+y_1+y_2=0,$$

$$xy+x_1y_1+x_2y_2=0.$$

eliminate x_1, x_2, y_1, y_2.

3. Examine the expression

$$(y-z)^n+(z-x)^n+(x-y)^n$$

for divisibility by

$$x^2+y^2+z^2-yz-zx-xy$$

for $n=1, 2, 3, 4, 5, 6$.

Denoting the given expression by P_n shew that

$$P_n . P_2 = 2P_{n+2} - \tfrac{2}{3} P_{n-1} . P_3,$$

or that

$$2P_3 . P_n + 3P_2 . P_{n+1} = 6P_{n+3}$$

and infer that P_n is divisible by $U(=x^2+y^2+z^2-yz-zx-xy)$ if n is of the form $6m-1$ or $6m+2$, is divisible by U^2 if n is of the form $6m+1$ or $6m-2$, and is not divisible by U if n is of the form $6m$ or $6m-3$.

4. Shew that every vulgar fraction may be expressed in a terminating series of the form

$$\frac{1}{q_1} - \frac{1}{q_1q_2} + \frac{1}{q_1q_2q_3} - \ldots\ldots$$

5. If a_r is the coefficient of x^r in the expansion of $\left(\frac{1+x}{1-x}\right)^n$, then

$$(r+1)a_{r+1} - 2na_r - (r-1)a_{r-1} = 0.$$

XX

1. Solve

$$\frac{x}{a}(xy + xz - 2yz) = \frac{y}{b}(yz + yx - 2zx) = \frac{z}{c}(zx + zy - 2xy) = k^3.$$

2. If a, b, x, y are rational numbers such that

$$(ay - bx)^2 + 4(a - x)(b - y) = 0$$

prove that either (i) $x = a$, $y = b$, or (ii) $1 - ab$ and $1 - xy$ are squares of rational numbers.

3. If

$$l^2 + m^2 + n^2 = 1, \qquad ll_1 + mm_1 + nn_1 = 0,$$

$$l_1^2 + m_1^2 + n_1^2 = 1, \qquad l_1l_2 + m_1m_2 + n_1n_2 = 0.$$

$$l_2^2 + m_2^2 + n_2^2 = 1, \qquad l_2l + m_2m + n_2n = 0,$$

shew that

$$l^2 + l_1^2 + l_2^2 = 1, \qquad lm + l_1m_1 + l^2m^2 = 0,$$

$$m^2 + m_1^2 + m_2^2 = 1, \qquad mn + m_1n_1 + m_2n_2 = 0,$$

$$n^2 + n_1^2 + n_2^2 = 1, \qquad nl + n_1l_1 + n_2l_2 = 0.$$

4. Shew that if n is a positive integer

$$\frac{1}{x+1} - \frac{n}{(x+1)(x+2)} + \frac{n(n-1)}{(x+1)(x+2)(x+3)} + \ldots\ldots$$

$$+ \frac{(-1)^n n!}{(x+1)(x+2)\ldots\ldots(x+n+1)} = \frac{1}{x+n+1}.$$

(To be established directly, and also by induction).

5. Find the number of triangles that can be formed with $2n$ lines of lengths $1, 2, \ldots\ldots 2n$.

INTEREST AND ANNUITY TABLES

$\frac{3}{4}\%$ $(i = 0{\cdot}0075,\ n = \text{number of years.})$

Years.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n=(1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\mid}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\mid}$ Amount of an annuity of 1 just after nth payment.	Years.
1	1·007500	0·992556	0·99256	1·00000	1
2	1·015056	0·985167	1·97772	2·00750	2
3	1·022669	0·977833	2·95556	3·02256	3
4	1·030339	0·970554	3·92611	4·04523	4
5	1·038067	0·963330	4·88944	5·07556	5
6	1·045852	0·956158	5·84560	6·11363	6
7	1·053696	0·949040	6·79464	7·15948	7
8	1·061599	0·941975	7·73661	8·21318	8
9	1·069561	0·934963	8·67158	9·27478	9
10	1·077583	0·928003	9·59958	10·34434	10
11	1·085664	0·921095	10·52067	11·42192	11
12	1·093807	0·914238	11·43491	12·50759	12
13	1·102010	0·907432	12·34235	13·60139	13
14	1·110276	0·900677	13·24302	14·70340	14
15	1·118603	0·893972	14·13700	15·81368	15
16	1·126992	0·887318	15·02431	16·93228	16
17	1·135445	0·880712	15·90503	18·05927	17
18	1·143960	0·874156	16·77918	19·19472	18
19	1·152540	0·867649	17·64683	20·33868	19
20	1·161184	0·861190	18·50802	21·49122	20
21	1·169893	0·854779	19·36280	22·65240	21
22	1·178667	0·848416	20·21121	23·82230	22
23	1·187507	0·842100	21·05331	25·00096	23
24	1·196414	0·835831	21·88915	26·18847	24
25	1·205387	0·829609	22·71876	27·38488	25
26	1·214427	0·823434	23·54219	28·59027	26
27	1·223535	0·817304	24·35949	29·80470	27
28	1·232712	0·811220	25·17071	31·02823	28
29	1·241957	0·805181	25·97589	32·26094	29
30	1·251272	0·799187	26·77508	33·50290	30
31	1·266056	0·793238	27·56832	34·75417	31
32	1·270111	0·787333	28·35565	36·01483	32
33	1·279637	0·781472	29·13712	37·28494	33
34	1·289234	0·775654	29·91278	38·56458	34
35	1·298904	0·769880	30·68266	39·85381	35
36	1·308645	0·764149	31·44680	41·15272	36
37	1·318460	0·758460	32·20527	42·46136	37
38	1·328349	0·752814	32·95808	43·77982	38
39	1·338311	0·747210	33·70529	45·10817	39
40	1·348349	0·741648	34·44694	46·44648	40

$\frac{7}{8}\%$ ($i = 0{\cdot}00875$, n = number of years.)

Years.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n = (1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\rceil}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\rceil}$ Amount of an annuity of 1 just after nth payment.	Years.
1	1·008750	0·991326	0·99133	1·00000	1
2	1·017577	0·982727	1·97405	2·00875	2
3	1·026480	0·974203	2·94826	3·02633	3
4	1·035462	0·965752	3·91401	4·05281	4
5	1·044522	0·957375	4·87138	5·08827	5
6	1·053662	0·949071	5·82045	6·13279	6
7	1·062881	0·940839	6·76129	7·18645	7
8	1·072182	0·932678	7·69397	8·24933	8
9	1·081563	0·924587	8·61856	9·32152	9
10	1·091027	0·916568	9·53513	10·40308	10
11	1·100573	0·908617	10·44374	11·49411	11
12	1·110203	0·900756	11·34448	12·59468	12
13	1·119918	0·892923	12·23740	13·70488	13
14	1·129717	0·885177	13·12258	14·82480	14
15	1·139602	0·877499	14·00008	15·95452	15
16	1·149574	0·869888	14·86997	17·09412	16
17	1·159632	0·862342	15·73231	18·24369	17
18	1·169779	0·854862	16·58717	19·40333	18
19	1·180015	0·847447	17·43462	20·57311	19
20	1·190340	0·840096	18·27471	21·75312	20
21	1·200755	0·832809	19·10752	22·94346	21
22	1·211262	0·825585	19·93311	24·14421	22
23	1·221860	0·818424	20·75153	25·25548	23
24	1·232552	0·811325	21·56286	26·57734	24
25	1·243337	0·804287	22·36715	27·80989	25
26	1·254216	0·797311	23·16446	29·05323	26
27	1·265190	0·790395	23·95485	30·30744	27
28	1·276261	0·783539	24·73839	31·57263	28
29	1·287428	0·776742	25·51513	32·84889	29
30	1·298693	0·770005	26·28514	34·13632	30
31	1·310056	0·763326	27·04847	35·43501	31
32	1·321519	0·756705	27·80517	36·74507	32
33	1·333083	0·750141	28·55531	38·06659	33
34	1·344747	0·743634	29·29895	39·39967	34
35	1·356514	0·737184	30·03613	40·74442	35
36	1·368383	0·730789	30·76692	42·10093	36
37	1·380357	0·724450	31·49137	43·46931	37
38	1·392435	0·718166	32·20954	44·84967	38
39	1·404618	0·711937	32·92147	46·24211	39
40	1·416909	0·705762	33·62723	47·64672	40

1% ($i = 0{\cdot}01$, n = number of years.)

YEARS.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n=(1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\mid}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\mid}$ Amount of an annuity of 1 just after nth payment.	YEARS.
1	1·010000	0·990099	0·99010	1·00000	1
2	1·020100	0·980296	1·97040	2·01000	2
3	1·030301	0·970590	2·94099	3 03010	3
4	1·040604	0·960980	3·90197	4·06040	4
5	1·051010	0·951466	4·85343	5·10101	5
6	1·061520	0·942045	5·79548	6·15202	6
7	1·072135	0·932718	6·72819	7·21354	7
8	1·082857	0·923483	7·65168	8 28567	8
9	1·093685	0·914340	8·56602	9·36853	9
10	1·104622	0·905287	9·47130	10·46221	10
11	1·115668	0·896324	10·36763	11·56683	11
12	1·126825	0·887449	11·25508	12·68250	12
13	1·138093	0·878663	12·13374	13·80933	13
14	1·149474	0 869963	13·00370	14·94742	14
15	1·160969	0·861349	13·86505	16·09690	15
16	1·172579	0·852821	14·71787	17·25786	16
17	1·184304	0·844377	15·56225	18·43044	17
18	1·196147	0·836017	16·39827	19·61475	18
19	1·208109	0·827740	17·22601	20·81089	19
20	1·220190	0·819544	18·04555	22·01900	20
21	1·232392	0·811430	18·85698	23·23919	21
22	1·244716	0·803396	19·66038	24·47159	22
23	1·257163	0·795442	20·45582	25·71630	23
24	1·269735	0·787566	21·24339	26·97346	24
25	1·282432	0·779768	22·02316	28·24320	25
26	1·295256	0·772048	22·79520	29·52563	26
27	1·308209	0·764404	23·55961	30·82089	27
28	1·321291	0·756836	24·31644	32·12910	28
29	1·334504	0·749342	25·06579	33·45039	29
30	1·347941	0·741923	25·80771	34·78489	30
31	1·361327	0·734577	26·54229	36 13274	31
32	1·374941	0·727304	27·26959	37·49407	32
33	1·388690	0·720103	27·98969	38·86901	33
34	1·402577	0·712973	28·70267	40·25770	34
35	1·416603	0·705914	29·40858	41·66028	35
36	1·430769	0·698925	30·10750	43 07688	36
37	1·445076	0·692005	30·79951	44·50765	37
38	1·459527	0·685153	31·48466	45·95272	38
39	1·474123	0·678370	32·16303	47·41225	39
40	1·488864	0·671653	32·83469	48·88637	40

$1\frac{1}{8}\%$ ($i = 0·01125$, n = number of years.)

Years.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n=(1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n\rceil}}$ Present value of an annuity of 1 running n years.	$s_{\overline{n\rceil}}$ Amount of an annuity of 1 just after nth payment.	Years.
1	1·011250	0·988875	0·98888	1·00000	1
2	1·022627	0·977874	1·96675	2·01125	2
3	1·034131	0·966995	2·93374	3·03388	3
4	1·045765	0·956238	3·88998	4·06801	4
5	1·057530	0·945600	4·83558	5·11377	5
6	1·069427	0·935080	5·77066	6·17130	6
7	1·081458	0·924677	6·69534	7·24073	7
8	1·093625	0·914391	7·60973	8·32219	8
9	1·105928	0·904218	8·51395	9·41581	9
10	1·118370	0·894159	9·40811	10·52174	10
11	1·130951	0·884211	10·29232	11·64011	11
12	1·143674	0·874375	11·16669	12·77106	12
13	1·156541	0·864647	12·03134	13·91474	13
14	1·169552	0·855028	12·88637	15·07128	14
15	1·182709	0·845516	13·73189	16·24083	15
16	1·196015	0·836110	14·56800	17·42354	16
17	1·209470	0·826808	15·39480	18·61955	17
18	1·223077	0·817610	16·21241	19·82902	18
19	1·236836	0·808515	17·02093	21·05210	19
20	1·250751	0·799520	17·82045	22·28894	20
21	1·264821	0·790625	18·61107	23·53969	21
22	1·279051	0·781830	19·39290	24·80451	22
23	1·293440	0·773132	20·16604	26·08356	23
24	1·307991	0·764531	20·93057	27·37700	24
25	1·322706	0·756026	21·68659	28·68499	25
26	1·337587	0·747615	22·43421	30·00770	26
27	1·352634	0·739298	23·17351	31·34528	27
28	1·367852	0·731073	23·90458	32·69792	28
29	1·383240	0·722940	24·62752	34·06577	29
30	1·398801	0·714898	25·34242	35·44901	30
31	1·414538	0·706945	26·04936	36·84781	31
32	1·430451	0·699080	26·74844	38·26235	32
33	1·446544	0·691303	27·43974	39·69280	33
34	1·462818	0·683612	28·12336	41·13934	34
35	1·479274	0·676007	28·79936	42·60216	35
36	1·495916	0·668487	29·46785	44·08143	36
37	1·512745	0·661050	30·12890	45·57735	37
38	1·529764	0·653696	30·78260	47·09010	38
39	1·546973	0·646424	31·42902	48·61986	39
40	1·564377	0·639232	32·06825	50·16683	40

$1\frac{1}{4}\%$ ($i = 0{\cdot}0125$, n = number of years.)

YEARS.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n = (1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\rvert}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\rvert}$ Amount of an annuity of 1 just after nth payment.	YEARS.
1	1·012500	0·987654	0·98765	1·00000	1
2	1·025156	0·975461	1·96312	2·02150	2
3	1·037971	0·963418	2·92653	3·03766	3
4	1·050945	0·951524	3·87806	4·07563	4
5	1·064082	0·939777	4·81783	5·12657	5
6	1·077383	0·928175	5·74601	6·19065	6
7	1·090950	0·916716	6·66273	7·26804	7
8	1·104486	0·905398	7·56812	8·35889	8
9	1·118292	0·894221	8·46234	9·46337	9
10	1·132271	0·883181	9·34553	10·58167	10
11	1·146424	0·872277	10·21780	11·71394	11
12	1·160755	0·861509	11·07931	12·86036	12
13	1·175264	0·850873	11·93018	14·02112	13
14	1·189955	0·840368	12·77055	15·19638	14
15	1·204829	0·829993	13·60055	16·38633	15
16	1·219890	0·819746	14·42029	17·59116	16
17	1·235138	0·809626	15·22992	18·81105	17
18	1·250577	0·799631	16·02955	20·04619	18
19	1·266210	0·789759	16·81931	21·29677	19
20	1·282037	0·780008	17·59932	22·56298	20
21	1·298063	0·770379	18·36969	23·84502	21
22	1·314288	0·760868	19·13056	25·14308	22
23	1·330717	0·751474	19·88204	26·45737	23
24	1·347351	0·742197	20·62423	27·78808	24
25	1·364193	0·733034	21·35727	29·13544	25
26	1·381245	0·723984	22·08125	30·49963	26
27	1·398511	0·715046	22·79630	31·88087	27
28	1·415992	0·706218	33·50252	33·27938	28
29	1·433692	0·697500	24·20002	34·69538	29
30	1·451613	0·688889	24·88891	36·12907	30
31	1·469759	0·680038	25·56929	37·58068	31
32	1·488131	0·671984	26·24127	39·05044	32
33	1·506732	0·663688	26·90496	40·53857	33
34	1·525566	0·655494	27·56046	42·04530	34
35	1·544636	0·647402	28·20786	43·57087	35
36	1·563944	0·639409	28·84727	45·11551	36
37	1·583493	0·631515	29·47878	46·67945	37
38	1·603287	0·623719	30·10250	48·26294	38
39	1·623328	0·616018	30·71852	49·86623	39
40	1·643619	0·608413	31·32693	51·48956	40

$1\frac{1}{2}\%$ ($i = 0·015$, n = number of years.)

YEARS.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n = (1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\vert}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\vert}$ Amount of an annuity of 1 just after nth payment.	YEARS.
1	1·015000	0·985222	0·98522	1·00000	1
2	1·030225	0·970662	1·95588	2·01500	2
3	1·045678	0·956317	2·91220	3·04523	3
4	1·061364	0·942184	3·85438	4·09090	4
5	1·077284	0·928260	4·78265	5·15227	5
6	1·093443	0·914542	5·69719	6·22955	6
7	1·109845	0·901027	6·59821	7·32299	7
8	1·126493	0·887711	7·48593	8·43284	8
9	1·143390	0·874592	8·36052	9·55933	9
10	1·160541	0·861667	9·22219	10·70272	10
11	1·177949	0·848933	10·07112	11·86326	11
12	1·195618	0·836387	10·90751	13·04121	12
13	1·213552	0·824027	11·73153	14·23683	13
14	1·231756	0·811849	12·54338	15·45038	14
15	1·250232	0·799851	13·34323	16·68214	15
16	1·268986	0·788031	14·13126	17·93237	16
17	1·288020	0·776385	14·90765	19·20136	17
18	1·307341	0·764912	15·67256	20·48938	18
19	1·326951	0·753607	16·42617	21·79672	19
20	1·346855	0·742470	17·16864	23·12367	20
21	1·367058	0·731498	17·90014	24·47052	21
22	1·387564	0·720688	18·62083	25·83758	22
23	1·408377	0·710037	19·33086	27·22514	23
24	1·429503	0·699544	20·03041	28·63352	24
25	1·450945	0·689206	20·71961	30·06302	25
26	1·472710	0·679020	21·39863	31·51397	26
27	1·494800	0·668986	22·06762	32·98668	27
28	1·517222	0·659099	22·72672	34·48148	28
29	1·539981	0·649359	23·37608	35·99870	29
30	1·563080	0·639762	24·01584	37·53868	30
31	1·586526	0·630308	24·64615	39·10176	31
32	1·610324	0·620993	25·26714	40·68829	32
33	1·634479	0·611816	25·87896	42·29861	33
34	1·658996	0·602774	26·48173	43·93309	34
35	1·683881	0·593866	27·07560	45·59209	35
36	1·709140	0·585090	27·66068	47·27597	36
37	1·734777	0·576443	28·23713	48·98511	37
38	1·760798	0·567924	28·80505	50·71989	38
39	1·787210	0·559531	29·36458	52·48068	39
40	1·814018	0·551262	29·91585	54·26789	40

$1\frac{3}{4}\%$ ($i = 0\cdot0175$, n = number of years.)

YEARS.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n=(1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\vert}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\vert}$ Amount of an annuity of 1 just after nth payment.	YEARS.
1	1·017500	0·982801	0·98280	1·00000	1
2	1·035306	0·965898	1·94870	2·01750	2
3	1·053424	0·949285	2·89798	3·05281	3
4	1·071850	0·932958	3·83094	4·10623	4
5	1·090617	0·916913	4·74786	5·17809	5
6	1·109702	0·901142	5·64900	6·26871	6
7	1·129122	0·885646	6·53464	7·37841	7
8	1·148882	0·870413	7·40505	8·50753	8
9	1·168987	0·855441	8·26049	9·65641	9
10	1·189444	0·840728	9·10122	10·82540	10
11	1·210260	0·826269	9·92749	12·01484	11
12	1·231439	0·812058	10·73955	13·22510	12
13	1·252990	0·798091	11·53764	14·45654	13
14	1·274917	0·784365	12·32201	15·70953	14
15	1·297228	0·770875	13·09288	16·98445	15
16	1·319929	0·757616	13·85050	18·28168	16
17	1·343028	0·744586	14·59508	19·60161	17
18	1·366531	0·731780	15·32686	20·94463	18
19	1·390445	0·719194	16·04606	22·31117	19
20	1·414778	0·706825	16·75288	23·70161	20
21	1·439537	0·694668	17·44755	25·11639	21
22	1·464729	0·682720	18·13027	26·55593	22
23	1·490361	0·670978	18·80125	28·02065	23
24	1·516443	0·659438	19·46069	29·51102	24
25	1·542981	0·648096	20·10878	31·02746	25
26	1·569983	0·636950	20·74573	32·57044	26
27	1·597457	0·625995	21·37173	34·14042	27
28	1·625413	0·615228	21·98695	35·73788	28
29	1·653858	0·604647	22·59160	37·36329	29
30	1·682800	0·594248	23·18585	39·01715	30
31	1·712249	0·584027	23·76988	40·69995	31
32	1·742213	0·573982	24·34386	42·41220	32
33	1·772702	0·564110	24·90797	44·15441	33
34	1·803725	0·554408	25·46238	45·92712	34
35	1·835290	0·544873	26·00725	47·73084	35
36	1·867407	0·535502	26·54275	49·56613	36
37	1·900087	0·526292	27·06904	51·43354	37
38	1·933338	0·517240	27·58628	53·33362	38
39	1·967172	0·508344	28·09463	55·26696	39
40	2·001597	0·499601	28·59423	57·23413	40

2% ($i = 0{\cdot}02$, n = number of years.)

Years.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n=(1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\rceil}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\rceil}$ Amount of an annuity of 1 just after nth payment.	Years.
1	1·020000	0·980392	0·98039	1·00000	1
2	1·040400	0·961169	1·94156	2·02000	2
3	1·061208	0·942322	2·88388	3·06040	3
4	1·082432	0·923845	3·80773	4·12161	4
5	1·104081	0·905731	4·71346	5·20404	5
6	1·126162	0·887971	5·60143	6·30812	6
7	1·148686	0·870560	6·47199	7·43428	7
8	1·171659	0·853490	7·32548	8·58297	8
9	1·195093	0·836755	8·16224	9·75463	9
10	1·218994	0·820348	8·98258	10·94972	10
11	1·243374	0·804263	9·78685	12·16872	11
12	1·268242	0·788493	10·57534	13·41209	12
13	1·293607	0·773032	11·34837	14·68033	13
14	1·319479	0·757875	12·10625	15·97394	14
15	1·345868	0·743015	12·84926	17·29342	15
16	1·372786	0·728446	13·57771	18·63929	16
17	1·400241	0·714163	14·29187	20·01207	17
18	1·428246	0·700159	14·99203	21·41231	18
19	1·456811	0·686431	15·67846	22·84056	19
20	1·485947	0·672971	16·35143	24·29737	20
21	1·515666	0·659776	17·01121	25·78332	21
22	1·545980	0·646839	17·65805	27·29898	22
23	1·576899	0·634156	18·29220	28·84496	23
24	1·608437	0·621721	18·91393	30·42186	24
25	1·640606	0·609531	19·52346	32·03030	25
26	1·673418	0·597579	20·12104	33·67091	26
27	1·706886	0·585862	20·70690	35·34432	27
28	1·741024	0·574375	21·28127	37·05121	28
29	1·775845	0·563112	21·84438	38·79223	29
30	1·811362	0·552071	22·39646	40·56808	30
31	1·847589	0·541246	22·93770	42·37944	31
32	1·884541	0·530633	23·46833	44·22703	32
33	1·922231	0·520229	23·98856	46·11157	33
34	1·960676	0·510028	24·49859	48·03380	34
35	1·999890	0·500028	24·99862	49·99448	35
36	2·039887	0·490223	25·48884	51·99437	36
37	2·080685	0·480611	25·96945	54·03425	37
38	2·122299	0·471187	26·44064	56·11494	38
39	2·164745	0·461948	26·90259	58·23724	39
40	2·208040	0·452890	27·35548	60·40198	40

$2\frac{1}{4}\%$ ($i = 0{\cdot}0225$, n = number of years.)

YEARS.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n=(1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\vert}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\vert}$ Amount of an annuity of 1 just after nth payment.	YEARS.
1	1·022500	0·977995	0·97799	1·00000	1
2	1·045506	0·956474	1·93447	2·02250	2
3	1·069030	0·935427	2·86990	3·06801	3
4	1·093083	0·914843	3·78474	4·13704	4
5	1·117678	0·894712	4·67945	5·23012	5
6	1·142825	0·875024	5·55448	6·34780	6
7	1·168539	0·855769	6·41025	7·49062	7
8	1·194831	0·836938	7·24718	8·65916	8
9	1·221715	0·818522	8·06571	9·85399	9
10	1·249203	0·800510	8·86622	11·07571	10
11	1·277311	0·782895	9·64911	12·32491	11
12	1·306050	0·765667	10·41478	13·60222	12
13	1·335436	0·748819	11·16360	14·90827	13
14	1·365483	0·732341	11·89594	16·24371	14
15	1·396207	0·716226	12·61217	17·60919	15
16	1·427621	0·700466	13·31263	19·00540	16
17	1·459743	0·685052	13·99768	20·43302	17
18	1·492587	0·669978	14·66766	21·89276	18
19	1·526170	0·655235	15·32290	23·38535	19
20	1·560509	0·640816	15·96371	24·91152	20
21	1·595621	0·626715	16·59043	26·47203	21
22	1·631522	0·612925	17·20335	28·06765	22
23	1·668231	0·599437	17·80279	29·69917	23
24	1·705767	0·586247	18·38904	31·36740	24
25	1·744146	0·573346	18·96238	33·07317	25
26	1·783390	0·560730	19·52311	34·81732	26
27	1·823516	0·548391	20·07150	36·60071	27
28	1·864545	0·536324	20·60783	38·42422	28
29	1·906497	0·524522	21·13235	40·28877	29
30	1·949393	0·512980	21·64533	42·19526	30
31	1·993255	0·501692	22·14702	44·14466	31
32	2·038103	0·490652	22·63767	46·13791	32
33	2·083960	0·479856	23·11753	48·17602	33
34	2·130849	0·469296	23·58683	50·25998	34
35	2·178794	0·458969	24·04580	52·39083	35
36	2·227816	0·448870	24·49467	54·56962	36
37	2·277942	0·438993	24·93366	56·79744	37
38	2·329196	0·429333	25·36299	59·07538	38
39	2·381603	0·419885	25·78288	61·40457	39
40	2·435189	0·410646	26·19352	63·78618	40

$2\frac{1}{2}\%$ ($i = 0{\cdot}025$, n = number of years.)

Years.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n = (1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\vert}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\vert}$ Amount of an annuity of 1 just after nth payment.	Years.
1	1·025000	0·975610	0·97561	1·00000	1
2	1·050625	0·951814	1·92742	2·02500	2
3	1·076891	0·928599	2·85602	3·07563	3
4	1·103813	0·905951	3·76197	4·15252	4
5	1·131408	0·883854	4·64583	5·25633	5
6	1·159693	0·862297	5·50813	6·38774	6
7	1·188686	0·841265	6·34939	7·54743	7
8	1·218403	0·820747	7·17014	8·73612	8
9	1·248863	0·800728	7·97087	9·95452	9
10	1·280085	0·781198	8·75206	11·20338	10
11	1·312087	0·762145	9·51421	12·48347	11
12	1·344889	0·743556	10·25776	13·79555	12
13	1·378511	0·725420	10·98318	15·14044	13
14	1·412974	0·707727	11·69091	16·51895	14
15	1·448298	0·690466	12·38138	17·93193	15
16	1·484506	0·673625	13·05500	19·38022	16
17	1·521618	0·657195	13·71220	20·86473	17
18	1·559659	0·641166	14·35336	22·38635	18
19	1·598650	0·625528	14·97889	23·94601	19
20	1·638616	0·610271	15·58916	25·54466	20
21	1·679582	0·595386	16·18455	27·18327	21
22	1·721571	0·580865	16·76541	28·86286	22
23	1·764611	0·566697	17·33211	30·58443	23
24	1·808726	0·552875	17·88499	32·34904	24
25	1·853944	0·539391	18·42438	34·15776	25
26	1·900293	0·526235	18·95061	36·01171	26
27	1·947800	0·513400	19·46401	37·91200	27
28	1·996495	0·500878	19·96489	39·85980	28
29	2·046407	0·488661	20·45355	41·85630	29
30	2·097568	0·476743	20·93029	43·90270	30
31	2·150007	0·465115	21·39541	46·00027	31
32	2·203757	0·453771	21·84918	48·15028	32
33	2·258851	0·442703	22·29188	50·35403	33
34	2·315322	0·431905	22·72379	52·61289	34
35	2·373205	0·421371	23·14516	54·92821	35
36	2·432535	0·411094	23·55625	57·30141	36
37	2·493349	0·401067	23·95732	59·73395	37
38	2·555682	0·391285	24·34860	62·22730	38
39	2·619574	0·381741	24·73034	64·78298	39
40	2·685064	0·372431	25·10277	67·40255	40

$2\frac{3}{4}\%$ ($i=0\cdot0275$, $n=$ number of years.)

Years.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n=(1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\mid}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\mid}$ Amount of an annuity of 1 just after nth payment.	Years.
1	1·027500	0·973236	0·97324	1·00000	1
2	1·055756	0·947188	1·92043	2·02750	2
3	1·084790	0·921838	2·84226	3·08326	3
4	1·114621	0·897166	3·73943	4·16805	4
5	1·145273	0·873154	4·61258	5·28267	5
6	1·176768	0·849785	5·46237	6·42794	6
7	1·209129	0·827041	6·28941	7·60471	7
8	1·242380	0·804906	7·09431	8·81384	8
9	1·276546	0·783364	7·87768	10·05622	9
10	1·311651	0·762398	8·64008	11·33276	10
11	1·347721	0·741993	9·38207	12·64442	11
12	1·384783	0·722134	10·10420	13·99214	12
13	1·422865	0·702807	10·80701	15·37692	13
14	1·461994	0·683997	11·49101	16·79979	14
15	1·502199	0·665691	12·15670	18·26178	15
16	1·543509	0·647874	12·80457	19·76398	16
17	1·585955	0·630535	13·43511	21·30749	17
18	1·629570	0·613659	14·04877	22·89344	18
19	1·674383	0·597235	14·64600	24·52301	19
20	1·720428	0·581251	15·22725	26·19740	20
21	1·767740	0·565694	15·79295	27·91783	21
22	1·816353	0·550554	16·34350	29·68557	22
23	1·866028	0·535819	16·87932	31·50192	23
24	1·917626	0·521478	17·40080	33·36822	24
25	1·970361	0·507521	17·90832	35·28585	25
26	2·024546	0·493938	18·40226	37·25621	26
27	2·080221	0·480718	18·88297	39·28075	27
28	2·137427	0·467852	19·35083	41·36098	28
29	2·196206	0·455331	19·80616	43·49840	29
30	2·256602	0·443144	20·24930	45·69461	30
31	2·318658	0·431284	20·68059	47·95121	31
32	2·382421	0·419741	21·10033	50·26987	32
33	2·447938	0·408507	21·50883	52·65229	33
34	2·515256	0·397574	21·90641	55·10023	34
35	2·584426	0·386933	22·24930	57·61548	35
36	2·655498	0·376577	22·66992	60·19991	36
37	2·728524	0·366499	23·03642	62·85541	37
38	2·803558	0·356690	23·39311	65·58393	38
39	2·880656	0·347144	23·74025	68·38749	39
40	2·959874	0·337852	24·07810	71·26814	40

3% ($i = 0{\cdot}03$, n = number of years.)

YEARS.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n = (1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\vert}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\vert}$ Amount of an annuity of 1 just after nth payment.	YEARS.
1	1·030000	0·970874	0·97087	1·00000	1
2	1·060900	0·942596	1·91347	2·03000	2
3	1·092727	0·915142	2·82861	3·09090	3
4	1·125509	0·888487	3·71710	4·18363	4
5	1·159274	0·862609	4·57971	5·30914	5
6	1·194052	0·837484	5·41719	6·46841	6
7	1·229874	0·813091	6·23028	7·66246	7
8	1·266770	0·789409	7·01969	8·89234	8
9	1·304773	0·766417	7·78611	10·15911	9
10	1·343916	0·744094	8·53020	11·46388	10
11	1·384234	0·722421	9·25262	12·80780	11
12	1·425761	0·701380	9·95400	14·19203	12
13	1·468534	0·680951	10·63496	15·61779	13
14	1·512590	0·661118	11·29607	17·08632	14
15	1·557967	0·641862	11·93794	18·59891	15
16	1·604706	0·623167	12·56110	20·15688	16
17	1·652848	0·605016	13·16612	21·76159	17
18	1·702433	0·587395	13·75351	23·41444	18
19	1·753506	0·570286	14·32380	25·11687	19
20	1·806111	0·553676	14·87748	26·87037	20
21	1·860295	0·537549	15·41502	28·67649	21
22	1·916103	0·521892	15·93692	30·53678	22
23	1·973589	0·506692	16·44361	32·45288	23
24	2·032794	0·491934	16·93554	34·42647	24
25	2·093778	0·477606	17·41315	36·45926	25
26	2·156591	0·463695	17·87684	38·55304	26
27	2·221289	0·450189	18·32703	40·70963	27
28	2·287928	0·437077	18·76411	42·93092	28
29	2·356566	0·424346	19·18846	45·21885	29
30	2·427262	0·411987	19·60044	47·57542	30
31	2·500080	0·399987	20·00043	50·00268	31
32	2·575083	0·388337	20·38877	52·50276	32
33	2·652335	0·377026	20·76579	55·07784	33
34	2·731905	0·366045	21·13184	57·73018	34
35	2·813862	0·355383	21·48722	60·46208	35
36	2·898278	0·345032	21·83225	63·27594	36
37	2·985227	0·334983	22·16724	66·17422	37
38	3·074783	0·325226	22·49246	69·15945	38
39	3·167027	0·315753	22·80822	72·23423	39
40	3·262038	0·306557	23·11477	75·40126	40

$3\frac{1}{4}\%$ ($i = 0·0325$, n = number of years.)

YEARS.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n = (1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\vert}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\vert}$ Amount of an annuity of 1 just after nth payment.	YEARS.
1	1·032500	0·968523	0·96852	1·00000	1
2	1·066056	0·938037	1·90656	2·03250	2
3	1·100703	0·908510	2·81507	3·09856	3
4	1·136475	0·879913	3·69498	4·19926	4
5	1·173411	0·852216	4·54720	5·33574	5
6	1·211547	0·825391	5·37259	6·50915	6
7	1·250923	0·799410	6·17200	7·72069	7
8	1·291578	0·774247	6·94625	8·97162	8
9	1·333554	0·749876	7·69612	10·26319	9
10	1·376894	0·726272	8·42240	11·59675	10
11	1·421643	0·703411	9·12581	12·97364	11
12	1·467848	0·681270	9·80708	14·39529	12
13	1·515552	0·659826	10·46690	15·86313	13
14	1·564807	0·639056	11·10596	17·37868	14
15	1·615663	0·618941	11·72490	18·94349	15
16	1·668173	0·599458	12·32436	20·55915	16
17	1·722388	0·580589	12·90495	22·22733	17
18	1·778366	0·562314	13·46726	23·94972	18
19	1·836163	0·544614	14·01187	25·72808	19
20	1·895838	0·527471	14·53935	27·56424	20
21	1·957453	0·510868	15·05021	29·46008	21
22	2·021070	0·494787	15·54500	31·41753	22
23	2·086755	0·479213	16·02421	33·43860	23
24	2·154574	0·464129	16·48834	35·52536	24
25	2·224598	0·449519	16·93786	37·67993	25
26	2·296897	0·435370	17·37323	39·90453	26
27	2·371546	0·421666	17·79490	42·20143	27
28	2·448622	0·408393	18·20329	44·57297	28
29	2·528202	0·395538	18·59883	47·02160	29
30	2·610368	0·383088	18·98192	49·54980	30
31	2·695205	0·371029	19·35295	52·16017	31
32	2·782780	0·359350	19·71230	54·85537	32
33	2·873241	0·348039	20·06034	57·63817	33
34	2·966621	0·337084	20·39742	60·51141	34
35	3·063036	0·326473	20·72389	63·47803	35
36	3·162585	0·316197	21·04009	66·54107	36
37	3·265369	0·306244	21·34633	69·70365	37
38	3·371493	0·296604	21·64294	72·96902	38
39	3·481067	0·287268	21·93021	76·34052	39
40	3·594201	0·278226	22·20843	79·82158	40

$3\frac{1}{2}\%$ ($i = 0{\cdot}035$, n = number of years.)

Years.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n = (1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\rvert}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\rvert}$ Amount of an annuity of 1 just after nth payment.	Years.
1	1·035000	0·966184	0·96618	1·00000	1
2	1·071225	0·933511	1·89969	2·03500	2
3	1·108718	0·901943	2·8[illegible]164	3·10623	3
4	1·147523	0·871442	3·67308	4·21494	4
5	1·187686	0·841973	4·51505	5·36247	5
6	1·229255	0·813501	5·32855	6·55015	6
7	1·272279	0·785991	6·11454	7·77941	7
8	1·316809	0·759412	6·87396	9·05169	8
9	1·362897	0·733731	7·60769	10·36850	9
10	1·410599	0·708919	8·31661	11·73139	10
11	1·459970	0·684946	9·00155	13·14199	11
12	1·511069	0·661783	9·66334	14·60196	12
13	1·563956	0·639404	10·30274	16·11303	13
14	1·618695	0·617782	10·92052	17·67699	14
15	1·675349	0·596891	11·51741	19·29568	15
16	1·733986	0·576706	12·09412	20·97103	16
17	1·794676	0·557204	12·65132	22·70502	17
18	1·857489	0·538361	13·18968	24·49969	18
19	1·922501	0·520156	13·70984	26·35718	19
20	1·989789	0·502566	14·21240	28·27968	20
21	2·059431	0·485571	14·69798	30·26947	21
22	2·131512	0·469151	15·16713	32·32890	22
23	2·206114	0·453286	15·62041	34·46041	23
24	2·283328	0·437957	16·05837	36·66653	24
25	2·363245	0·423147	16·48152	38·94986	25
26	2·445959	0·408838	16·89035	41·31310	26
27	2·531567	0·395012	17·28537	43·75906	27
28	2·620172	0·381654	17·66702	46·29063	28
29	2·711878	0·368748	18·03577	48·91080	29
30	2·806794	0·356278	18·39205	51·62268	30
31	2·905031	0·344230	18·73628	54·42947	31
32	3·006708	0·332590	19·06887	57·33450	32
33	3·111942	0·321343	19·39021	60·34121	33
34	3·220860	0·310476	19·70069	63·45315	34
35	3·333590	0·299977	20·00066	66·67401	35
36	3·450266	0·289833	20·29050	70·00760	36
37	3·571025	0·280032	20·57053	73·45787	37
38	3·696011	0·270562	20·84109	77·02889	38
39	3·825372	0·261412	21·10250	80·72491	39
40	3·959260	0·252572	21·35507	84·55028	40

4% ($i=0·04$, n = number of years.)

Years.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n=(1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\rvert}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\rvert}$ Amount of an annuity of 1 just after nth payment.	Years.
1	1·040000	0·961538	0·96154	1·00000	1
2	1·081600	0·924556	1·88609	2·04000	2
3	1·124864	0·888996	2·77509	3·12160	3
4	1·169859	0·854804	3·62990	4·24646	4
5	1·216653	0·821927	4·45182	5·41632	5
6	1·265319	0·790314	5·24214	6·63298	6
7	1·315932	0·759918	6·00205	7·89829	7
8	1·368569	0·730690	6·73275	9·21423	8
9	1·423312	0·702587	7·43533	10·58280	9
10	1·480244	0·675564	8·11090	12·00611	10
11	1·539454	0·649581	8·76048	13·48635	11
12	1·601032	0·624597	9·38507	15·02581	12
13	1·665074	0·600574	9·98565	16·62684	13
14	1·731676	0·577475	10·56312	18·29191	14
15	1·800944	0·555264	11·11839	20·02359	15
16	1·872981	0·533908	11·65230	21·82453	16
17	1·947900	0·513373	12·16567	23·69751	17
18	2·025817	0·493628	12·65930	25·64541	18
19	2·106849	0·474642	13·13394	27·67123	19
20	2·191123	0·456387	13·59033	29·77808	20
21	2·278768	0·438834	14·02916	31·96920	21
22	2·369919	0·421955	14·45112	34·24797	22
23	2·464716	0·405726	14·85684	36·61789	23
24	2·563304	0·390121	15·24696	39·08260	24
25	2·665836	0·375117	15·62208	41·64591	25
26	2·772470	0·360689	15·98277	44·31174	26
27	2·883369	0·346817	16·32959	47·08421	27
28	2·998703	0·333477	16·66306	49·96758	28
29	3·118651	0·320651	16·98372	52·96629	29
30	3·243398	0·308319	17·29203	56·08494	30
31	3·373133	0·296460	17·58849	59·32834	31
32	3·508059	0·285058	17·87355	62·70147	32
33	3·648381	0·274094	18·14765	66·20953	33
34	3·794316	0·263552	18·41120	69·85791	34
35	3·946089	0·253415	18·66461	73·65222	35
36	4·103933	0·243669	18·90828	77·59831	36
37	4·268090	0·234297	19·14258	81·70225	37
38	4·438813	0·225285	19·36787	85·97034	38
39	4·616366	0·216621	19·58449	90·40915	39
40	4·801021	0·208289	19·79277	95·02552	40

$4\frac{1}{2}\%$ ($i=0{\cdot}045$, $n=$ number of years.)

YEARS.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n=(1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\|}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\|}$ Amount of an annuity of 1 just after nth payment.	YEARS.
1	1·045000	0·956938	0·95694	1·00000	1
2	1·092025	0·915730	1·87267	2·04500	2
3	1·141166	0·876297	2·74896	3·13703	3
4	1·192519	0·838561	3·58753	4·27819	4
5	1·246182	0·802451	4·38998	5·47071	5
6	1·302260	0·767896	5·15787	6·71689	6
7	1·360862	0·734828	5·89270	8·01915	7
8	1·422101	0·703185	6·59589	9·38001	8
9	1·486095	0·672904	7·26879	10·80211	9
10	1·552969	0·643928	7·91272	12·28821	10
11	1·622853	0·616199	8·52892	13·84118	11
12	1·695881	0·589664	9·11858	15·46403	12
13	1·772196	0·564272	9·68285	17·15991	13
14	1·851945	0·539973	10·22283	18·93211	14
15	1·935282	0·516720	10·73955	20·78405	15
16	2·022370	0·494469	11·23401	22·71934	16
17	2·113377	0·473176	11·70719	24·74171	17
18	2·208479	0·452800	12·15999	26·85508	18
19	2·307860	0·433302	12·59329	29·06356	19
20	2·411714	0·414643	13·00794	31·37142	20
21	2·520241	0·396787	13·40472	33·78314	21
22	2·633652	0·379701	13·78442	36·30338	22
23	2·752166	0·363350	14·14777	38·93703	23
24	2·876014	0·347703	14·49548	41·68920	24
25	3·005434	0·332731	14·82821	44·56521	25
26	3·140679	0·318402	15·14661	47·57064	26
27	3·282010	0·304691	15·45130	50·71132	27
28	3·429700	0·291571	15·74287	53·99333	28
29	3·584036	0·279015	16·02189	57·42303	29
30	3·745318	0·267000	16·28889	61·00707	30
31	3·913857	0·255502	16·54439	64·75239	31
32	4·089981	0·244500	16·78889	68·66625	32
33	4·274030	0·233971	17·02286	72·75623	33
34	4·466362	0·223896	17·24676	77·03026	34
35	4·667348	0·214254	17·46101	81·49662	35
36	4·877378	0·205028	17·66604	86·16397	36
37	5·096860	0·196199	17·86224	91·04134	37
38	5·326219	0·187750	18·04999	96·13820	38
39	5·565899	0·179665	18·22966	101·46442	39
40	5·816365	0·171929	18·40158	107·03032	40

5% ($i = 0{\cdot}05$, n = number of years.)

Years.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n=(1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\rvert}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\rvert}$ Amount of an annuity of 1 just after nth payment.	Years.
1	1·050000	0·952381	0·95238	1·00000	1
2	1·102500	0·907029	1·85941	2·05000	2
3	1·157625	0·863838	2·72325	3·15250	3
4	1·215506	0·822702	3·54595	4·31013	4
5	1·276282	0·783526	4·32948	5·52563	5
6	1·340096	0·746215	5·07569	6·80191	6
7	1·407100	0·710681	5·78637	8·14201	7
8	1·477455	0·676839	6·46321	9·54911	8
9	1·551328	0·644609	7·10782	11·02656	9
10	1·628895	0·613913	7·72173	12·57789	10
11	1·710339	0·584679	8·30641	14·20679	11
12	1·795856	0·556837	8·86325	15·91713	12
13	1·885649	0·530321	9·39357	17·71298	13
14	1·979932	0·505068	9·89864	19·59863	14
15	2·078928	0·481017	10·37966	21·57856	15
16	2·182875	0·458111	10·83777	23·65749	16
17	2·292018	0·436297	11·27407	25·84037	17
18	2·406619	0·415521	11·68959	28·13238	18
19	2·526950	0·395734	12·08532	30·53900	19
20	2·653298	0·376889	12·46221	33·06595	20
21	2·785963	0·358942	12·82115	35·71925	21
22	2·925261	0·341849	13·16300	38·50521	22
23	3·071524	0·325571	13·48857	41·43048	23
24	3·225100	0·310068	13·79864	44·50200	24
25	3·386355	0·295303	14·09394	47·72710	25
26	3·555673	0·281241	14·37518	51·11345	26
27	3·733456	0·267848	14·64303	54·66913	27
28	3·920129	0·255094	14·89813	58·40258	28
29	4·116136	0·242946	15·14107	62·32271	29
30	4·321942	0·231377	15·37245	66·43885	30
31	4·538039	0·220359	15·59281	70·76079	31
32	4·764941	0·209866	15·80268	75·29883	32
33	5·003189	0·199872	16·00255	80·06377	33
34	5·253348	0·190355	16·19290	85·06696	34
35	5·516015	0·181290	16·37419	90·32031	35
36	5·791816	0·172657	16·54685	95·83632	36
37	6·081407	0·164436	16·71129	101·62814	37
38	6·385477	0·156605	16·86789	107·70955	38
39	6·704751	0·149148	17·01704	114·09502	39
40	7·039989	0·142046	17·15909	120·79977	40

$5\frac{1}{2}\%$ ($i = 0\,055$, n = number of years.)

Years.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n = (1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\vert}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\vert}$ Amount of an annuity of 1 just after nth payment.	Years.
1	1·055000	0·947867	0·94787	1·00000	1
2	1·113025	0·898452	1·84632	2·05500	2
3	1·174241	0·851613	2·69793	3·16803	3
4	1·238825	0·807217	3·50515	4·34227	4
5	1·306960	0·765134	4·27028	5·58109	5
6	1·378843	0·725246	4·99553	6·88805	6
7	1·454679	0·687437	5·68297	8·26689	7
8	1·534686	0·651599	6·33457	9·72157	8
9	1·619094	0·617629	6·95220	11·25626	9
10	1·708144	0·585430	7·53763	12·87535	10
11	1·802092	0·554910	8·09254	14·58350	11
12	1·901207	0·525981	8·61852	16·38559	12
13	2·005774	0·498561	9·11708	18·28680	13
14	2·116091	0·472569	9·58965	20·29257	14
15	2·232476	0·447933	10·03758	22·40866	15
16	2·355263	0·424581	10·46216	24·64114	16
17	2·484802	0·402446	10·86461	26·99640	17
18	2·621466	0·381466	11·24607	29·48120	18
19	2·765647	0·361579	11·60765	32·10267	19
20	2·917757	0·342728	11·95038	34·86832	20
21	3·078234	0·324861	12·27524	37·78608	21
22	3·247537	0·307926	12·58317	40·86431	22
23	3·426152	0·291873	12·87504	44·11185	23
24	3·614590	0·276656	13·15170	47·53800	24
25	3·813392	0·262234	13·41393	51·15259	25
26	4·023129	0·248563	13·66250	54·96598	26
27	4·244401	0·235604	13·89810	58·98911	27
28	4·477843	0·223322	14·12142	63·23351	28
29	4·724124	0·211679	14·33310	67·71135	29
30	4·983951	0·200644	14·53375	72·43548	30
31	5·258069	0·190183	14·72393	77·41943	31
32	5·547262	0·180269	14·90420	82·67750	32
33	5·852362	0·170871	15·07507	88·22476	33
34	6·174242	0·161963	15·23703	94·07712	34
35	6·513825	0·153520	15·39055	100·25136	35
36	6·872085	0·145516	15·53607	106·76519	36
37	7·250050	0·137930	15·67400	113·63727	37
38	7·648803	0·130739	15·80474	120·88732	38
39	8·069487	0·123924	15·92866	128·53613	39
40	8·513309	0·117463	16·04612	136·60561	40

6% ($i=0·06$, $n=$ number of years.)

YEARS.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n=(1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\vert}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\vert}$ Amount of an annuity of 1 just after nth payment.	YEARS.
1	1·060000	0·943396	0·94340	1·00000	1
2	1·123600	0·889996	1·83339	2·06000	2
3	1·191016	0·839619	2·67301	3·18360	3
4	1·262477	0·792094	3·46511	4·37462	4
5	1·338226	0·747258	4·21236	5·63709	5
6	1·418519	0·704960	4·91732	6·97532	6
7	1·503630	0·665057	5·58238	8·39384	7
8	1·593848	0·627412	6·20979	9·89747	8
9	1·689479	0·591898	6·80169	11·49132	9
10	1·790848	0·558395	7·36009	13·18079	10
11	1·898299	0·526787	7·88687	14·97164	11
12	2·012196	0·496969	8·38384	16·86994	12
13	2·132928	0·468839	8·85268	18·88214	13
14	2·260904	0·442301	9·29498	21·01507	14
15	2·396558	0·417265	9·71225	23·27597	15
16	2·540352	0·393646	10·10590	25·67253	16
17	2·692773	0·371364	10·47726	28·21288	17
18	2·854339	0·350344	10·82760	30·90565	18
19	3·025599	0·330513	11·15812	33·75999	19
20	3·207135	0·311805	11·46992	36·78559	20
21	3·399564	0·294155	11·76408	39·99273	21
22	3·603537	0·277505	12·04158	43·39229	22
23	3·819750	0·261797	12·30338	46·99583	23
24	4·048935	0·246978	12·55036	50·81558	24
25	4·291871	0·232999	12·78336	54·86451	25
26	4·549383	0·219810	13·00317	59·15638	26
27	4·822346	0·207368	13·21053	63·70577	27
28	5·111687	0·195630	13·40616	68·52811	28
29	5·418388	0·184557	13·59072	73·63980	29
30	5·743491	0·174110	13·76483	79·05819	30
31	6·088101	0·164255	13·92909	84·80168	31
32	6·453387	0·154957	14·08404	90·88978	32
33	6·840590	0·146186	14·23023	97·34316	33
34	7·251025	0·137911	14·36814	104·18375	34
35	7·686087	0·130105	14·49825	111·43478	35
36	8·147252	0·122741	14·62099	119·12087	36
37	8·636087	0·115793	14·73678	127·26812	37
38	9·154252	0·109239	14·84602	135·90421	38
39	9·703507	0·103055	14·94907	145·05846	39
40	10·285718	0·097222	15·04630	154·76197	40

7% ($i=0{\cdot}07$, $n=$ number of years.)

Years.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n=(1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\vert}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\vert}$ Amount of an annuity of 1 just after nth payment.	Years.
1	1·070000	0·934579	0·93458	1·00000	1
2	1·144900	0·873439	1·80802	2·07000	2
3	1·255043	0·816298	2·62432	3·21490	3
4	1·310796	0·762895	3·38721	4·43994	4
5	1·402552	0·712986	4·10020	5·75074	5
6	1·500730	0·666342	4·76654	7·15329	6
7	1·605781	0·662750	5·38929	8·65402	7
8	1·718186	0·582009	5·97130	10·25980	8
9	1·838459	0·543934	6·51523	11·97799	9
10	1·967151	0·508349	7·02358	13·81645	10
11	2·104852	0·475093	7·49867	15·78360	11
12	2·522192	0·444012	7·94269	17·88845	12
13	2·409845	0·414964	8·35765	20·14064	13
14	2·578534	0·387817	8·74547	22·55049	14
15	2·759032	0·362446	9·10791	25·12902	15
16	2·952164	0·338735	9·44665	27·88805	16
17	3·158815	0·316574	9·76322	30·84022	17
18	3·379932	0·295864	10·05909	33·99903	18
19	3·616528	0·276508	10·33560	37·37896	19
20	3·869684	0·258419	10·59401	40·99549	20
21	4·140562	0·241513	10·83553	44·86518	21
22	4·430402	0·225713	11·06124	49·00574	22
23	4·740530	0·210947	11·27219	53·43614	23
24	5·072367	0·197147	11·46933	58·17667	24
25	5·427433	0·184249	11·65358	63·24904	25
26	5·807353	0·172195	11·82578	68·67647	26
27	6·213868	0·160930	11·98671	74·48382	27
28	6·648838	0·150402	12·13711	80·69769	28
29	7·114257	0·140563	12·27767	87·34653	29
30	7·612255	0·131367	12·40904	94·46079	30
31	8·145113	0·122773	12·53181	102·07304	31
32	8·715271	0·114741	12·64656	110·21815	32
33	9·325340	0·107235	12·75379	118·93343	33
34	9·978114	0·100219	12·85401	128·25876	34
35	10·676581	0·093663	12·94767	138·23688	35
36	11·423942	0·087535	13·03521	148·91346	36
37	12·223618	0·081809	13·11702	160·33740	37
38	13·079271	0·076457	13·19347	172·56102	38
39	13·994820	0·071455	13·26493	185·64029	39
40	14·974458	0·066780	13·33171	199·63511	40

8% ($i=0{\cdot}08$, $n=$ number of years.)

YEARS.	$(1+i)^n$ Amount of 1 at end of n years.	$v^n=(1+i)^{-n}$ Present value of 1 due at the end of n years.	$a_{\overline{n}\rvert}$ Present value of an annuity of 1 running n years.	$s_{\overline{n}\rvert}$ Amount of an annuity of 1 just after nth payment.	YEARS.
1	1·080000	0·925926	0·92593	1·00000	1
2	1·166400	0·857339	1·78326	2·08000	2
3	1·259712	0·793832	2·57710	3·24640	3
4	1·360489	0·735030	3·31213	4·50611	4
5	1·469328	0·680583	3·99271	5·86660	5
6	1·586874	0·630170	4·62288	7·33593	6
7	1·713824	0·583490	5·20637	8·92280	7
8	1·850930	0·540269	5·74664	10·63663	8
9	1·999005	0·500249	6·24689	12·48756	9
10	2·158925	0·463193	6·71008	14·48656	10
11	2·331639	0·428883	7·13896	16·64549	11
12	2·518170	0·397114	7·53608	18·97713	12
13	2·719624	0·367698	7·90378	21·49530	13
14	2·937194	0·340461	8·24424	24·21492	14
15	3·172169	0·315242	8·55948	27·15211	15
16	3·425943	0·291890	8·85137	30·32428	16
17	3·700018	0·270269	9·12164	33·75023	17
18	3·996019	0·250249	9·37189	37·45024	18
19	4·315701	0·231712	9·60360	41·44626	19
20	4·660957	0·214548	9·81815	45·76196	20
21	5·033834	0·198656	10·01680	50·42292	21
22	5·436540	0·183940	10·20074	55·45676	22
23	5·871464	0·170315	10·37106	60·89330	23
24	6·341181	0·157699	10·52876	66·76476	24
25	6·848475	0·146018	10·67478	73·10594	25
26	7·396353	0·135202	10·80998	79·95442	26
27	7·988061	0·125187	10·93516	87·35077	27
28	8·627106	0·115914	11·05108	95·33883	28
29	9·317275	0·107327	11·15841	103·96594	29
30	10·062657	0·099377	11·25778	113·28321	30
31	10·867669	0·092016	11·34980	123·34587	31
32	11·737083	0·085200	11·43500	134·21354	32
33	12·676050	0·078889	11·51389	145·95062	33
34	13·690134	0·073045	11·58693	158·62667	34
35	14·785344	0·067634	11·65457	172·31680	35
36	15·968172	0·062625	11·71719	187·10215	36
37	17·245626	0·057986	11·77518	203·07032	37
38	18·625276	0·053690	11·82887	220·31595	38
39	20·115298	0·049713	11·87858	238·94122	39
40	21·724521	0·046031	11·92461	259·05652	40

ANSWERS

Exercises : Page 5

1. (1) $x(x+1)-x^2=13$; (2) $3x+15=2(x+15)$; (3) $2(100-x\times 100\div 10)=100-x\times 100\div 11$. **2.** (1) A property of every three consecutive integers; (2) A property for every division of the line.

Exercises : Page 6

1. (1) 11; (2) 12; (3) pqr; (4) abc. **2.** 15 ft., 27 ft.

Exercises : Page 9

1. $0, x=0$; $-1, x=0$; $5\frac{3}{4}, x=\frac{1}{2}$; $2, x=1$. **2.** $-5\frac{3}{4}, x=\frac{1}{2}$; $0, x=0$; $1, x=0$; $4, x=1$. **3.** $-6\frac{1}{8}$, $x=\frac{3}{4}$, min.; $4\frac{1}{12}$, $x=\frac{5}{6}$, min.; $7\frac{1}{8}$, $x=1\frac{1}{4}$, max.

Exercises : Page 15

1. $\left\{x+(7+\sqrt{69})\div 2\right\}\left\{x+(7-\sqrt{69})\div 2\right\}$; $(x-1)(2x-5)$; $\left\{2x+(\sqrt{73}+5)\div 4\right\}\left\{(\sqrt{73}-5)\div 4-2x\right\}$; $7\left\{x-(11+\sqrt{-19})\div 14\right\}\left\{x-(11-\sqrt{-19})\div 14\right\}$. **2.** $(7\pm\sqrt{37})\div 2$; $(2\pm\sqrt{19})\div 3$; $(-3\pm\sqrt{149})\div 10$; $(11\pm\sqrt{-131})\div 6$. **3.** Real and positive; real, one positive and one negative, the former the greater numerically; real and negative; real and positive. **4.** $x^2-9x+11=0$. **5.** $ax^2+(b-2ah)x+c-bh+ah^2=0$. **6.** $x^2-10x+12=0$. **7.** $x^2+mpx+m^2q=0$. **9.** (1) $4x^2-37x+9=0$; (2) $3x^2-7x+2=0$; (3) $8x^2-42x+27=0$. **10.** (1) $a^2x^2-(b^2-2ac)x+c^2=0$; (2) $cx^2+bx+a=0$; (3) $c^2x^2-(b^2-2ac)x+a^2=0$; (4) $a^3x^2+b(b^2-3ac)x+c^3=0$; (5) $a^3x^2+abcx+c^3=0$; (6) $acx^2-(b^2-4ac)x+ac=0$. **11.** $(q-s)=(p-r)(rq-ps)$. **12.** $a\div p=b\div q=c\div r$. **14.** Zero for $x=4$ or -3; negative for x between 4 and -3; positive for other values. **15.** Always positive. **16.** $a(\sqrt{5}-1)\div 2$ and $a(3-\sqrt{5})\div 2$; $a(3+\sqrt{5})\div 2$ and $a(-1-\sqrt{5})\div 2$.

Exercises : Page 21

1. $\pm\sqrt{6}\div 2$, $\pm\sqrt{35}\div 5$. **2.** $\frac{7}{5}$, $\frac{5}{7}$, $(-1\pm\sqrt{-3})\div 2$. **3.** 4, -13, $(9\pm 3\sqrt{-31})\div 2$. **4.** 5, $-\frac{5}{2}$, $\left\{(5\pm\sqrt{137}\div 4\right\}$. **5.** 2, 2, $(9\pm\sqrt{-175})\div 8$. **6.** $\frac{2}{3}$, $\frac{3}{2}$, $(-3\pm\sqrt{-7})\div 4$. **7.** 1, -1, $+\sqrt{-1}$, $-\sqrt{-1}$. **8.** 3, -10, $(-7\pm 3\sqrt{-11})\div 2$. **9.** 7, -28. **10.** 7, -13, $\left\{-3\pm\sqrt{79}\right\}$.

Exercises : Page 24

1. 3, 5. **2.** 1, -1. **3.** 36, 12. **4.** 1, 2, 3. **5.** $\frac{354}{41}, -\frac{120}{41}, -\frac{23}{41}$.

Exercises : Page 27

1. 4, 3 ; $9\frac{3}{7}$, $-2\frac{3}{7}$. **2.** 5, 3 ; 3, 5. **3.** ± 3, ∓ 7 ; $\pm 5\sqrt{-5}$, $\mp 2\sqrt{-5}$. **4.** ± 2, ∓ 1 ; $\pm\frac{46}{25}$, $\mp\frac{1}{25}$. **5.** 5, 4 ; 4, 5. **6.** ± 2, ∓ 1; ± 1, ∓ 2. **7.** 2, 3 ; $\frac{234}{107}$, $\frac{306}{107}$. **8.** ± 7, ∓ 6 ; ± 6, ∓ 7. **9.** 9, 12 ; 0, 0. **10.** $\frac{5}{2}, \frac{3}{2}$; $\frac{3}{2}, \frac{5}{2}$; $(-5\pm\sqrt{-35})\div 4$, $(-5\mp\sqrt{-35})\div 4$. **11.** 2, 3 ; 3, 2 ; 5, 1 ; 1, 5. **12.** ± 2, ± 3 ; $\pm 8i$, $\mp 5i$. **13.** 0, 0 ; 0, 0 ; $\frac{12}{7}$, $\frac{4}{7}$; -4, 2. **14.** 7, 4 ; -4, -7. **15.** 3, 4 ; 4, 3 ; $(-7\pm\sqrt{97})\div 2$, $(-7\mp\sqrt{97})\div 2$.

Examples : Page 43

1. 7 : 4 or 5 : 3. **2.** -3. **3.** 7. **4.** $ab : a'b'$. **5.** 30. **17.** ± 3, ± 5, ± 11. **18.** 0, 0, 0 ; 3, 2, 1. **19.** 2, 3, 4.

Exercises : Page 51

1. $a=\frac{22}{7}r^2$. **2.** $a=\frac{88}{7}r^2$. **3.** $r=6$. **4.** 8·16 ft. **5.** 223·15 days.

Exercises : Page 56

1. $v=\frac{1}{3}\,bh$. **2.** $v=3{\cdot}1416\ r^2h$; 197·9208. **3.** 26·59g. **4.** (1) $y=2$; (2) $x=1{\cdot}5$.

Examples : Page 57

1. $v^2=64{\cdot}4s$. **2.** 0·0622 *ohms*. **3.** $v=\frac{1}{3}\,bh$. **5.** x and y sides of a right-angled triangle remaining similar to itself. **6.** $x=14$. **7.** $y=12$. **8.** $y=1$. **9.** $y=10{\cdot}5$. **10.** 1·693 *m*. **11.** 3·98125 ft. **12.** $x : y :: 3 : 4$ or $:: 4 : 3$.

Exercises : Page 62

1. 7257, 4058, 210222, 10011010101. **2.** ·2, ·24062.., ·124141..., ·259148036. **3.** e5·6, 3042·4, 636·205602..., 632·1515... **4.** 9. **5.** 7.

Exercises : Page 64

1. 12740, 2e08, 5562006, 2t6, with remainder 135. **2.** 1604, 5281617, 11t33, 42638652. **3.** 33·43, 2·35, 62·4753, 15·9... **4.** 21054, 1734, 18952, 2241. **6.** 23, 347, 122, 123. **7.** 5. **8.** $8(x-3)^4+89(x-3)^3+380(x-3)^2+753(x-3)+575$.

Exercises : Page 69

1. (1) 93, $2n-1$; (2) 283, $6n+1$; (3) -155, $33-4n$; (4) -504, $107-13n$. **2.** $53+48+43+\ldots\ldots$; -57, -2. **3.** 3, 12, 21. **4.** 30. **11.** 69. **13.** 0.

Exercises: Page 72

1. 36, 53, ..., 206. 2. 13, 9, ..., -3. 3. $\left\{(n-r+1)a+rb\right\}\div(n+1)$. 4. $13\frac{1}{3}$, $15\frac{2}{3}$, ..., $22\frac{2}{3}$.

Exercises: Page 74

1. (1) 8533; (2) $1510\frac{1}{2}$; (3) -689; (4) 53; (5) $(53a+1378b)\div b$. 2. (1) $2n^2+7n$; (2) $\frac{1}{2}(51n-5n^2)$; (3) $\frac{1}{8}(5n^2+15n)$. 5. 4 or 8. 6. 5. 7. 8. 8. 7. 9. 13. 10. $\frac{1}{2}(5r^2+19r)$. 11. $-52\frac{1}{7}-40\frac{4}{7}-29-\ldots\ldots$; 2405. 13. 10, 465. 14. $14r+4$. 15. 5, 8, 11. 16. $29+35+41+\ldots\ldots$; 14. 17. $-17-15-13\ldots\ldots$ 19. $16{\cdot}1t^2$.

Exercises: Page 78

1. (1) 5^8, 5^{n-1}; (2) 7.2^8, 7.2^{n-1}; (3) $2\div3^8$, $2\div3^{n-1}$; (4) $1\div2^8$, $1\div2^{n-1}$; (5) $1\div2^8$, $(-1)^{n-1}\div2^{n-1}$. 2. $a=480$, $r=\frac{1}{2}$; $480\div2^{16}$. 3. 12, 24, 48. 4. $r=27$. 9. $1\frac{1}{3}$. 10. $r=b^2$.

Exercises: Page 80

1. 21, 63, 189. 2. 35, 245..... 3. 24, 144.....

Exercises: Page 82

1. (1) $\frac{1}{2}(5^{29}-1)$; (2) $\frac{1}{6}(7-7^{-28})$; (3) $\frac{1}{3}(2+2^{-28})$; (4) $\frac{2}{5}+\frac{2}{5}(\frac{2}{3})^{29}$; (5) $a(1-a^{2n})\div(1-a^2)$. 2. (1) 4^n-1; (2) $12-12\div2^n$; (3) $\frac{3}{2}-(-1)^n\frac{1}{2}\div3^{n-1}$. 3. $(x^n-y^n)\div(x-y)$. 5. $n+2(x^n-1)\div x^n(x-1)+(x^{2n}-1)\div x^{2n}(x^2-1)$. 6. 9.12.16. 7. 17, 51, 153; $\frac{425}{3}$, $-\frac{595}{3}$, $\frac{833}{3}$. 8. 48, 72, 108. 10. 1st term $h(r-1)$; ratio r. 12. $\frac{10}{9}(10^n-1)-n$. 13. $\left\{a^2(a^n-1)\div(a-1)-b^2(b^n-1)\div(b-1)\right\}\div(a-b)$.

Exercises: Page 87

1. (1) $\frac{3}{2}$; (2) 3; (3) $\frac{2}{3}$. 2. (1) 15; (2) $3\frac{1}{2}$; (3) 21. 5. $r=\frac{4}{7}$. 6. $\frac{7}{9}$, $\frac{13}{999}$, $\frac{73146}{99900}$. 7. Area of original square.

Exercises: Page 90

1. 105, -105, -35, -21. 2. 6 and infinity. 3. 840, 420, 280.

Exercises: Page 92

1. 12, 15. 2. $7\frac{1}{2}$, 6, $4\frac{4}{5}$. 5. $pq\div(p+q)$.

Exercises : Page 94

1. $n(4n^2-1)\div 3$. 2. $n(n+1)(n+2)\div 3$. 3. $2n(n+1)(2n+1)\div 3$. 4. $n(4n^2+17n+21)\div 2$. 5. $na^2+n(n-1)ab+n(n-1)(2n-1)b^2\div 6$.

Exercises : Page 96

1. $n^2(2n^2-1)$. 2. $n(n+1)(n+2)(n+3)\div 4$. 3. $2n^2(n+1)^2$. 4. $n(n+1)(9n^2+23n+13)\div 6$. 5. $na^3+\frac{1}{2}n(n+1).3a^2b+\frac{1}{2}n(n+1)(2n+1)ab^2+\frac{1}{4}n^2(n+1)^2b^3$.

Exercises : Page 97

1. $(1-x^n)\div(1-x)^2-nx^n\div(1-x)$. 2. $6-1\div 2^{n-3}-(2n-1)\div 2^{n-1}$. 3. $7\div(1-x)+5x(1-x^{n-1})\div(1-x)^2-(5n+2)x^n\div(1-x)$. 4. Change x to $-x$ in (1.).

Examples : Page 97

1. (1) 84, 96, 108 ; (2) $93\frac{3}{4}$, $117\frac{3}{16}$, $146\frac{31}{64}$; (3) 120, 240, infinity. 2. 48, 80. 3. 9, 15, 21, 27. 5. (1) No number if a, b, c are not in A.P.; (2) $(ac-b^2)\div(2b-a-c)$; (3) $(2ac-ab-bc)\div(2b-a-c)$. 7. $5n(n+1)$. 8. $n(n-1)(2n-1)\div 3$. 9. (1) $n-\frac{1}{9}+\frac{1}{9}\div 10^n$; (2) $7n\div 9-7\div 81+7\div(81.10^n)$. 10. $r(r^2+1)\div 2$. 11. $a(1-r^n)\div(1-r)$ where $r^{n+1}=b\div a$. 13. (1) $(1-r^n)\div(1-b)(1-r)-b(1-b^nr^n)\div(1-b)(1-br)$; (2) $x(1-x^n)\div(1-x)+\frac{1}{2}an(n+1)$; (3) $n(n+1)(6n^2-2n-1)\div 6$. 14. $(1-x)^{-2}+2x(1-x^{n-1})(1-x)^{-3}-(n^2+2n-1)x^n(1-x)^{-2}+n^2x^{n+1}(1-x)^{-2}$. 18. $b+c(2r-1)$; an A.P. except for the first term. 19. $4+12+20+\ldots\ldots$ 20. $(a\div r-b\div r^2)(r^{2n}-1)\div(r^2-1)r^{2n-2}$

Exercises : Page 117

1. (1) 3, 3 ; (2) 3, 6 ; (3) 1, 6. 2. (1) 4, 4 ; (2) 6, 12 ; (3) 4, 24 ; (4) 1, 24.

Exercises : Page 119

1. 120. 2. 14820309504000 ; 3991680 ; 40320. 3. 840. 4. 3024 ; 1680 ; 336. 5. 4536 ; 504. 6. 6561. 7. 259. 8. 144. 9. 28800. 10. (1) 362880 ; (2) 40320 ; (3) 20160. 11. 2903040. 12. 2880.

Exercises : Page 121

1. 324632. 2. 35, 126, 1. 3. 455. 4. 18. 5. 35. 6. $n(n-1)(n-2)\div 3!$ 7. 2204475. 8. 14, 90. 9. 21627587520 ; 152922336. 11. 756756. 12. 126126. 13. 25200. 14. 6300. 16. $\{n(n-1)(n-2)-p(p-1)(p-2)\}\div 3!$ 18. $n(n-1)(n-2)(n-3)\div 8$. 19. 8. 20. 30. 22. 7200 ; 4320.

EXAMPLES : PAGE 125

1. (1) 15120; (2) 59049. 210, 1680; 729, 6561. **2.** $(p+1)(q+1)-1$. **3.** 1470, 89. **5.** $(m+n-2)!\div\{(m-1)!\,.\,(n-1)!\}$. **6.** 800. **7.** $(p+1)!\div\{n!\,(p-n+1)!\}$. **8.** (1) 495; (2) 35; (3) 10.

EXERCISES : PAGE 128

3. 10, 16, 18.

EXERCISES : PAGE 130

2. $6!\ x^3y^3\div(3!\ 3!)$; $20!\ x^{10}y^{10}\div(10!\ 10!)$; $-14!\ (2a)^7(3y)^7\div(7!\ 7!)$; $77x^6y^6\div3888$. 3. $5!\ (2a)^3(3b)^2\div(3!\ 2!)$, $-5!\ (2a)^2(3b)^3\div2!\ 3!$; $-23!\ x^{12}y^{11}\div(12!\ 11!)$, $23!\ x^{11}y^{12}\div(11!\ 12!)$; $17!\ x^{18}y^{16}\div9!\ 8!$, $-17!\ x^{16}y^{18}\div8!\ 9!$. 4. $(-1)^rn!\ x^r\div(n-r)!\ r!$; $m!\ 3^rx^r\div(m-r)!\ r!$; $(-1)^rn!\ a^{n-r}2^rx^r\div(n-r)!\ r!$

EXERCISES : PAGE 136

1. (1) x^r; (2) $(-1)^rx^r$; (3) $(r+1)x^r$; (4) $(r+1)(r+2)x^r\div1.2$; (5) $(r+1)(r+2)(r+3)x^r\div1.2.3$; (6) $m(m+1).\ldots.(m+r-1)x^r\div1.2.3.\ldots.r$. **2.** $-1.3.\ldots.(2r-3)x^r\div(2.4.\ldots.2r)$; $1.3.\ldots.(2r-1)x^r\div(2.4.\ldots.2r)$; $(-1)^r 1.3.\ldots.(2r-5)3a^{\frac{3}{2}}x^r\div(2.4.\ldots.2r.a^r)$; $5.6.\ldots.(r+4)(2a)^{-5}3^rx^r\div(1.2.\ldots.r.2^ra^r)$. **3.** $-9.7.5.3x^6\div2.4.\ldots.12$. **5.** $(-1)^{r-1}2.5.\ldots.(3r-4)x^r\div3.6.\ldots.3r$; $(-1)^r2.5.\ldots.(3r-1)x^r\div r!$; $1.6.11.\ldots.(5r-4)x^r\div(5.10.\ldots.5r)$. **6.** $(r+1)^2$; $2r^2+2r+1$; $8r-4$. **8.** $(-1)^nn(n+1).\ldots.(n+r-1)\div r!$. **10.** (1) $(1-\frac{1}{3})^{-2}$; (2) $(1+\frac{1}{2})^{-\frac{1}{2}}$; (3) $(1-\frac{2}{3})^{-\frac{1}{2}}$; (4) $(1-\frac{3}{5})^{-\frac{1}{3}}$.

EXERCISES : PAGE 138

1. 4·89898; 8·94427; 24·8998; 14·9666; 34·9285. **2.** 3·97906; 9·84886; 0·14586; 2·99256; 0·198945. **3.** 0·000026. **4.** $1-x$; $\frac{1}{2}+\frac{23}{16}x$; $\frac{1}{2}+\frac{41}{336}x$; $1-\frac{1}{2}x$; $\frac{14}{45}-\frac{413}{675}x$.

EXERCISES : PAGE 144

1. (1) 6th and 7th; (2) 1st; (3) 4th, 6th and 7th; (4) 3rd and 4th. **2.** (1) 5th and 6th; (2) 8th; (3) 24th and 25th; (4) 19th.

EXAMPLES : PAGE 145

1. $(-1)^n(2n)!\div(n!\ n!)$. **5.** 2^n; $2^n(n+r+1)-n2^{n-1}$. **6.** (1) $\sqrt{5}\div2$; $3^{\frac{1}{3}}\div2^{\frac{1}{3}}-1$. **7.** $2^n(n+r)-n2^{n-1}$. **10.** $x^n\div(1-x)$; $\{(n+1)x^n-nx^{n+1}\}\div(1-x)^2$. **12.** $25!\div(5!\,.\,7!\,.\,13!)$. **14.** $n(n-1)(4n^2+16n-21)\div6$.

Miscellaneous Examples : Page 157

I. 2. $(3x-5)(2x+3)$, $(3x-5y)(2x-7y)$, $(x+4+\sqrt{7})(x+4-\sqrt{7})$, $[x+(5\div 2+\sqrt{13}\div 2)y][x+(5\div 2-\sqrt{13}\div 2)y]$, $2(x-3\div 4+\sqrt{65}\div 4)(x-3\div 4-\sqrt{65}\div 4)$, $[2x-(11\div 4-\sqrt{41}\div 4)y]\,[2x-(11\div 4+\sqrt{41}\div 4)y]$. 3. (1) 2023344; (2) 160185; (3) 3011000000011; (4) $2etz6$. 4. 3, -10, $(-7\pm\sqrt{-159})\div 2$.

II. 3. (1) 14316; (2) 1152; (3) 116112; (4) 4523. 4. 1, $-6\frac{1}{2}$, $(-11\pm\sqrt{-199})\div 4$.

III. 2. $(x+2y+3)(3x+5y+8)$, $(2x-3y+4z)(3x+4y-7z)$. 3. (1) 149378; (2) $5z{\cdot}tzt$; (3) $21ete9$; (4) $e96{\cdot}18$; $48{\cdot}z36$; (5) $8336\frac{8}{9}$; $53{\cdot}7$. 4. 1, $-11\div 2$, $(-9\pm\sqrt{-159})\div 4$.

IV. 4. -2, $-\frac{1}{2}$, $(19\pm\sqrt{-215})\div 24$.

V. 2. $(1-ax)(1-ax-cx^2)$; $(x-c)(ax+by)(ax-by)$, $(a+d)(a^2+bc+ca+ab)$. 3. 23, 56, 123, 356. 4. 2, -12, $-4\pm\sqrt{-42}$.

VI. 3. 7. 4. $(5, 3)$, $(-5, -3)$, $(3i, -5i)$, $(-3i, 5i)$.

VII. 1. p^2-2q, p^3-3pq, $p^4-4p^2q+2q^2$, $p^5-5p^3q+5pq^2$. 2. $(x^2+xy+y^2)(x^2-xy+y^2)$; $(x^2-a^2-ab-b^2)(x^2+a^2-ab+b^2)$; $(1-x)(1+x)(1+x+y-xy)(1-x+y+xy)$; $(a^2+b^2)(x^2+y^2)(b^2-a^2)(x^2-y^2)$. 4. $(5, 4)$, $(1, 8)$.

VIII. 4. $(3, 2)$, $(-17\div 9, -34\div 27)$, $[(45\pm\sqrt{33849})\div 78,\ 45\pm\sqrt{33849})\div 104]$.

IX. 2. $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$, $(b+c+d-a)(c+d+a-b)(d+a+b-c)(a+b+c-d)$. 4. $(\frac{1}{2}, \frac{1}{3})$, $(\frac{1}{3}, \frac{1}{2})$, $(-1, \frac{1}{6})$, $(\frac{1}{6}, -1)$.

X. 4. $[(m\pm\sqrt{m^2-12})\div 2,\ (m\mp\sqrt{m^2-12})\div 2]$ where $m=(7+\sqrt{17})\div 2$.

XI. 2. (1) $(c-b)(a-c)(b-a)$; (2) $3(y+z)(z+x)(x+y)$; (3) $3(y-z)(z-x)(x-y)$. 4. $\frac{2}{3}$, $\frac{3}{2}$, -2, $-\frac{1}{2}$.

XII. 4. $(m\pm\sqrt{m^2-4})\div 2$ where $m=(1\pm\sqrt{5})\div 2$.

XIII. 2. (1) $(c-b)(a-c)(b-a)(a+b+c)$; (2) $12xyz(x+y+z)$. 4. 2, $-\frac{1}{2}$, $\frac{3}{2}$, $-\frac{2}{3}$.

XIV. 1. $ax^2+2brx+r^2c=0$. 2. (1) $(c-b)(a-c)(b-a)(bc+ca+ab)$; (2) $(c-b)(a-c)(b-a)(a^2+b^2+c^2+bc+ca+ab)$. 3. Those whose denominators are of the form 2^m. 3^n. 4. 4, $-\frac{1}{4}$, 2, $-\frac{1}{2}$.

XV. 2. (1) $(b+c)(c+a)(a+b)$; (2) $(c-b)(a-c)(b-a)(a+b+c)$; (3) $80xyz(x^2+y^2+z^2)$. 4. 4, $-\frac{10}{3}$, $(1\pm\sqrt{166})\div 3$; the last two values satisfy the equation only if the square root appearing in it is taken negatively.

XVI. 3. $w=2\sqrt{s^3}\div 9\sqrt{6}$. 4. (3, 4), (4, 3), $(-6\pm 2\sqrt{6},\ -6\mp\sqrt{6})$.

XVII. 3. (1) $-4n$; (2) $8n+1$; $-n(-1)^n$. 4. (2, 3), (3, 2), (1, 5), (5, 1).

XVIII. 3. $y=2x+x^2$. 4. (2, 3), $(-2, -3)$, $(\sqrt{6}, \sqrt{6})$, $(-\sqrt{6}, -\sqrt{6})$.

XIX. 2. $7\div 3$. 4. (3, 2), $(3\omega, 2\omega)$, $(3\omega^2, 2\omega^2)$, (2, 3), $(2\omega, 3\omega)$, $(2\omega, 3\omega^2)$. 5. $y:z= -amn\pm l\sqrt{(-bcl^2-cam^2-abn^2)} : am^2+bl^2$, etc.

XX. 3. $3:2$. 4. (4, 1), (1, 4), $[(5\pm\sqrt{-159})\div 2,\ (5\mp\sqrt{-159})\div 2]$. 5. $pb^2-2qab+a^2r=0$.

XXI. 1. (1) $(x-a)(x-b)=0$; (2) $(x-a)^2+(y-b)^2+(z-c)^2=0$. 2. $(3\pm\sqrt{5})\div 2$. 3. 13. 4. $(\sqrt{bc}, \sqrt{ca}, \sqrt{ab})$, $(\sqrt{bc}, -\sqrt{ca}, -\sqrt{ab})$, $(-\sqrt{bc}, \sqrt{ca}, -\sqrt{ab})$, $(-\sqrt{bc}, -\sqrt{ca}, \sqrt{ab})$.

XXII. 5. $4(a'b-ab')(b'c-bc')-(ac'-a'c)^2=0$.

XXIII. 1. $a=1$, $b=2$, $c=-3$; $\dfrac{1}{x+1}+\dfrac{2}{x+2}-\dfrac{3}{x+3}$. 4. (0, 0, 0), (4, 5, 7). 5. $\dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{c}{c'}$.

XXIV. 3. $\sqrt{5}+\sqrt{3}$, $\sqrt{6}-\sqrt{5}$, $\sqrt{(\frac{15}{2})}+\sqrt{(\frac{1}{2})}$. 4. (0, 0, 0), (2, 1, 3). 5. $27(bc'-b'c)(ab'-a'b)^2-(ca'-c'a)^3=0$.

XXV. 1. $\dfrac{2}{x-3}-\dfrac{3}{x-4}+\dfrac{5}{x-6}$, $2+\dfrac{2}{x+3}+\dfrac{1}{x-1}$, $\dfrac{1}{(a-b)(a-c)}\cdot\dfrac{1}{x-a}+\dots\dots+\dots\dots$ 3. $d=3$. 4. (0, 0, 0), $[a^2(b^2+c^2-a^2)l,\ b^2(c^2+a^2-b^2)l,\ c^2(a^2+b^2-c^2)l]$, where $l^2=2\div(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)$.

XXVI. 3. $3+2i$, $4-3i$. 4. (2, 3), (3, 2), $[(5\pm\sqrt{41})\div 2,\ (5\mp\sqrt{41})\div 2]$. 5. $(bc'-b'c)^2(ab'-a'b)-(ca'-c'a)^3=0$.

XXVII. 4. (9, 3), (3, 9), $[(13\pm\sqrt{-39})\div 2,\ (13\mp\sqrt{-39})\div 2]$. 5. $x^2+y^2=a^2+b^2$.

XXVIII. 1. $x^3+2px^2+p^2x+q^2=0$. 4. (3, 2), (2, 3), $(4\pm\sqrt{10},\ 4\mp\sqrt{10})$. 5. $k^4(b^2c^2-a^2d^2)^2+abc^2d^2(ab^2-ad^2-2k^2b)(ba^2-bc^2-2k^2a)=0$.

XXIX. 4. (2, 3), (3, 2), $[(-5\pm\sqrt{89})\div 2,\ (-5\mp\sqrt{89})\div 2]$.

XXX. 1. $x^2+(p-2h)x+h^2-ph+q=0$.

XXXI. 4. (7, 9, 11), (−7, −9, −11), $[(\pm 27 \pm 2\sqrt{-3}) \div 3$, etc., etc.].

XXXII. 1. $\frac{2}{5}$, $\frac{43}{13}$; for $\frac{43}{13} > x > \frac{2}{5}$; for $x > \frac{43}{13}$ and $x < \frac{2}{5}$; min. is $-35721 \div 260$. 4. (2, 3, 4), (−2, −3, −4).

XXXIII. 4. $x : y : z : 1 : : a^2(b^2+c^2) : b^2(c^2+a^2) : c^2(a^2+b^2) : \pm 2abc$.

XXXIV. 1. $1 > x > -\frac{5}{2}$; max. $= 245 \div 100$. 4. $x - y + z = \pm\sqrt{(a^2 - b^2 + c^2)}$, etc.

XXXV. 3. $n^2 - n + 1$; n^3. 4. (3, 4, 5), (−3, −4, −5).

XXXVI. 4. (1) $6! - 2 . 5!$; 144.

XXXVII. 3. 1296, 784. 5. 495; 1680.

XXXVIII. 1. $(a+b)^2 \div 8$, for $x = (a+b) \div 4$. 3. 2 values; 3 values; 6 values. 4. 44; 77. 5. $[2^{2n} - (2n)! \div (n! . n!)] \div 2$.

XXXIX. 3. (1) $2n(n+1)(2n+1) + 17n(n+1) \div 2 + 6n$. (2) $60[n(n+1) \div 2]^2 + 145n(n+1)(2n+1) \div 6 + 115n(n+1) \div 2 + 30n$.

XL. 1. $\infty > m > 3$. 2. (1) 27 for $x = -2$; $3 > x > -7$. (2) $3\frac{1}{8}$ for $x = \frac{1}{4}$; $\frac{3}{2} > x > -1$. (3) $-\frac{1}{4}$ for $x = \frac{5}{2}$; $3 > x > 2$. (4) $-6\frac{1}{8}$ for $x = 3\frac{1}{4}$; $5 > x > 1\frac{1}{2}$. (5) $a^2 \div 2$ for $x = a \div \sqrt{2}$; $x^2 < a^2$. 3. $x^3 + y^3 + z^3 + 3y^2z + 3yz^2 + 3z^2x + 3zx^2 + 3x^2y + 3xy^2 - 21xyz = 0$. 4. $12! \div (3!.3!.3!.3!.4!)$. 5. $1 + \frac{5}{3}x + \frac{23}{9}x^2 + \frac{255}{9}x^3$; $1 + 3x + \frac{13}{2}x^2 + \frac{21}{2}x^3$.

XLI. 1. (1) $2x+3$; (2) $A\dfrac{x-b}{a-b} + B\dfrac{x-a}{b-a}$. 2. $2a^2$. 3. $n^4 \div 2 - n(n+1)(2n+1) \div 3 + n(n+1) - n \div 2$. 4. $p^2(am - hl)^2 = (ap^2 - cl^2)(am^2 - 2hlm + bl^2)$.

XLII. 1. 1, for $x = -1$. 2. The isosceles triangle. 3. 800. 4. $(3n)! \div (n! . n! . x! . 3!)$.

XLIII. 1. (1) $\frac{1}{2}x^2 - 3\frac{1}{2}x + 4$; (2) $l\dfrac{(x-b)(x-c)}{(a-b)(a-c)} + m\dfrac{(x-c)(x-a)}{(b-c)(b-a)} + n\dfrac{(x-a)(x-b)}{(c-a)(c-b)}$. 2. When $x = y$. 3. (1) $4\frac{1}{2} - (4n-3) \div 2.3^{n-1} - 1 \div 3^{n-2}$. (2) $x \div (r-1) - (x + ny - y) \div r^n(r-1) + y \div (r-1)^2 - y \div r^{n-2}(r-1)^2$.

XLIV. 1. max. $= 3$, min. $= \frac{1}{3}$. 2. When the perpendicular is drawn from the mid-point of the base. 4. $\dfrac{n(n-1)(n-2)(n-3)(n-4)(2n-1)}{2 . 3! . 3!}$. 5. $3n$.

XLV. 2. When the triangle is isosceles. 4. $(b^2 - ac) \div a^2 = (b'^2 - a'c') \div a'^2$. 5. Coefficient of x^r.

XLVI. **1.** (1) max. $=1\frac{1}{9}$, min. $=3$. (A maximum is greater than neighboring values of the function ; it may be less than a minimum of the function. The graphic representation exhibits this fact very strikingly). (2) max. $=1\frac{1}{3}$, min. $=2$. **3.** 10 per cent. **4.** Remove restriction one person in each room ; 4^{10}. **5.** $2^{\frac{2}{3}}$.

XLVII. **1.** $A=\frac{1}{3}$, $B=\frac{1}{2}$, $C=\frac{1}{6}$, $D=0$. **2.** The geometrical centre. **4.** $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$.

XLVIII. **2.** 24 sq. in. **3.** $(3, 5)$, $(3\omega, 5\omega)$, $(3\omega^2, 5\omega^2)$. **4.** $18\,!\div[(2\,!)^3 . (3\,!)^4 . 3\,! . 4\,!]$. **5.** $3\sqrt{3}$.

XLIX. **1.** $A=2$, $B=2$, $C=1$, $D=-1$, $E=0$. **5.** $(r-1)(r-2) \ldots\ldots (r-m+1) \div (m-1)\,!$, for $r=\frac{1}{2}$, $r=-\frac{3}{2}$, $r=-\frac{2}{3}$.

L. **1.** $mn\,(\sqrt{2}-1)$ where m and n measure the sides AB and BC. **4.** $(p+2n)\,! \div [p\,! . n\,! . n\,!\ 2\,!]$.

LI. **1.** $f^2=bc$, $g^2=ca$, $h^2=ab$. **2.** 2 ft. **4.** $abc-a^2-b^2-c^2+2=0$. **5.** $(\sqrt{29}+5)^{2n}+(\sqrt{29}-5)^{2n}$ is the first integer in excess.

LII. **2.** That with edges all equal.

LIII. **2.** The cube. **4.** $l^4+m^4-n^2l^2-n^2m^2=0$. **5.** $(n+1)^2$.

LIV. **4.** $(m+n)\,!\div(m\,!\ n\,!)$.

LV. **2.** $12\frac{19}{27}$, $9\frac{13}{27}$. **4.** $(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)=8d^6$.

LVI. **2.** $\frac{4}{3\sqrt{3}}\pi r^3$. **4.** 55. **5.** $\frac{x^n}{1-x}$, $\frac{(n+1)x^n-nx^{n+1}}{(1-x)^2}$, $\left[\frac{n^2+3n+2}{1.2}x^n - n(n+2)x^{n+1} - \frac{n(n+1)x^{n+2}}{1.2}\right]\div(1-x)^3$.

LVII. **2.** $\frac{32}{81}\pi r^3$. **4.** $a^2+b^2+c^2-abc-4=0$. **5.** $(n+1)(n+2)(n+3)(n+4)(n+5)\div 60$.

LVIII. **4.** $(m+1)(m+2)\div 2$. **5.** The third.

LIX. **2.** When the radius of the base is two-thirds that of the base of the cone. **4.** $a^2+b^2+c^2-abc-4=0$.

LX. **2.** When the radius of the base is one-half that of the base of the cone.

ADDITIONAL MISCELLANEOUS EXAMPLES: PAGE 205

I. 1. 0, 4, $2 \pm \sqrt{-3}$. 4. $abc + 2fgh - af^2 - by^2 - ch^2 = 0$ (Read $2hxy$). 5. (i) $(m+n)! \div (m!\,.\,n!)$; (ii) $(n+1)^m$.

II. 1. $(-3, -2)$, $(-\frac{1}{3}, -\frac{1}{2})$.

III. 5. $2(m-1)!\,(n-1)! \div [(r-1)!\,(r-1)!\,(m-r)!\,(m-r)!]$.

IV. 1. $x-a=y-b=z-c=-(a+b+c)\div 3$. 3. Max. 4 for $x=2$, min. 10 for $x=8$. 5. Coefficient of x^{m-1} in $(1-x)^{n-2}$.

V. 1. ∞. 3. The cone whose height is double the diameter of the sphere. 5. $n(n+1)(n+2)(n+3)\div 12$.

VI. 1. One root becomes ∞, the other becomes $-c\div 2b$. 4. 5758π ft.

VII. 1. Take $xy + \frac{1}{xy} = u$, $\frac{x}{y} + \frac{y}{x} = 0$. 5. (1) Twice coefficient of x^{n-1} in $(1+x)^n(1-x)^{-3}$. (2) $2^n n^2$.

VIII. 1. $(ax+by)(ax+cz)=a^4$, etc.

IX. 4. $-8m^2$, $8m^2+8m+1$, $-(-1)^n 2n^2 - \frac{1}{2} + (-1)^n\frac{1}{2}$. 5. Sum of coefficients from that of x^{3m} on in expansion of $(1+x+x^2+\ldots\ldots+x^{4m})^3$.

X. 2. Square, and see lvii, Ex. 4. 3. (1) $4\frac{1}{27}$; (2) 4.

XI. See lix, Ex. 4.

XII. 1. $(x+2y+z)(x+y+2z)=a^2$, etc. 3. Find max. of $(x-1)$. $m(x-2).n(7-x)$ where the multipliers are to make the factors equal and sum constant. Then $1+m-n=0$, $m=(x-1)\div(x-2)$, $n=(x-1)\div(7-x)$; whence $(x-2)(7-x)+(x-1)(7-x)-(x-2)(x-1)=0$; this gives the values of x which make the function a maximum or a minimum.

XIII. 4. 24; 6.

XIV. 4. $\frac{1}{x} - \frac{n}{1}.\frac{1}{x+1} + \frac{n(n-1)}{1.2}\frac{1}{x+2} -$ etc. 5. $[a^{n+2}(c-b)+\ldots\ldots+\ldots\ldots]\div[(b-c)(c-a)(a-b)]$.

XV. 2. $(a^3+b^3+c^3+abc)(a^3+b^3+c^3-3abc)=0$. 3. The cylinder with height equal to radius of base. 5. $n(n-1)(n-2)(n-3)\div 24$; $n(n-3)(n-4)(n-5)\div 12$.

XVI. **2.** $a(p^2-qr)+b(q^2-rp)+c(r^2-pq)=0$. **3.** Distance of chord from centre is $r\div\sqrt{2}$. Discuss the case in which the given point is nearer the centre than this distance. **4.** $\frac{1}{4}[2n+1-(-1)^n]$.

XVII. **1.** $x : a^4-b^2c^2 :: y : b^4-c^2a^2 :: z : c^4-a^2b^2$.

XVIII. **3.** (*a*) $2a^3\div 27$; (*b*) If sides are $2a$ and $2b$ max. corresponds to a cutting of side $x=(a+b-\sqrt{a^2+b^2-ab})\div 3$. **5.** $n(n+1)\div 2+1$.

XIX. **2.** $a^2x^2+b^2y^2=\frac{2}{3}ma^2b^2$.

XX. $n(n-1)(4n-5)\div 6$.